Exercise 1:
Prove the soundness of the following rule, called the Differential Cut:

\[
\Gamma \vdash [\dot{x} = e \& Q] C \quad \Gamma \vdash [\dot{x} = e \& Q \land C] P
\]

\[
\Gamma \vdash [\dot{x} = e \& Q] P
\]

(DC)

where C is any first order formula of the real arithmetic.

Exercise 2:
In this exercise, we propose to find the invariant

\[2gz \leq 2gH - v^2\]

used in the bouncing ball example, using a template-based method. For this exercise, we assume that \(H, c\) and \(g\) are constants such that \(0 < c \leq 1\) and \(g > 0\). We want to find values of \(\alpha, \beta, \gamma, \delta, \sigma\) and \(\zeta\) such that the function

\[f(z, v) = \alpha z^2 + \beta zv + \gamma v^2 + \delta z + \sigma v + \zeta\]

can be used as an invariant to prove that \(z \leq H\).

1. We have seen that, to prove that \(f(z, v) \geq 0\) is an invariant of the dynamics \(e = (\dot{z} = v, \dot{v} = -g)\), it is enough to prove that the Lie derivative \(L_e f\) is the zero function. Compute this Lie derivative and deduce that, for this Lie derivative to be the zero function, it is enough to have:

\[\alpha = \beta = \sigma = 0 \quad \text{and} \quad \delta = 2\gamma g.\]

2. Now, we assume that \(f\) is of the form

\[f(z, v) = \gamma v^2 + 2\gamma gz + \zeta.\]

We have to prove that \(f(z, v) \geq 0\) is preserved by the jump function of the bouncing event, that is:

\[f(z, v) \geq 0 \Rightarrow f(z, -cv) \geq 0.\]

Prove that this implication holds if \(\gamma \leq 0\).

3. Next, we have to prove that \(z \leq H\) is implied by the invariant, that is:

\[f(z, v) \geq 0 \Rightarrow z \leq H.\]

Prove that this implication holds if

\[\gamma < 0 \quad \text{and} \quad -\gamma v^2 \geq 2\gamma gH + \zeta\]

for all \(v\).

Deduce that the latter inequality holds if \(\zeta = -2\gamma gH\).

4. Summarising all the conditions so far, we have \(f\) of the form:

\[f(z, v) = \gamma v^2 + 2\gamma gz - 2\gamma gH\]

with \(\gamma < 0\). To conclude, it is enough to prove that \(f(z, v) \geq 0\) holds initially. Prove it.