Deductive Verification Of Hybrid Systems

Home assignment

Your solution must be sent by email at dubut[at]nii[dot]ac[dot]jp before August 9th, 8pm. Any request can be made by email too.

Exercise 1 :

Prove the soundness of the following rule, called the Differential Cut :

$$\frac{\Gamma \vdash [\dot{\mathbf{x}} = \mathbf{e} \;\&\; Q]\, C \qquad \Gamma \vdash [\dot{\mathbf{x}} = \mathbf{e} \;\&\; Q \wedge C]\, P}{\Gamma \vdash [\dot{\mathbf{x}} = \mathbf{e} \;\&\; Q]\, P}\,\textbf{(DC)}$$

where C is any first order formula of the real arithmetic.

Exercise 2 :

In this exercise, we propose to find the invariant

$$2gz \leq 2gH - v^2$$

used in the bouncing ball example, using a template-based method. For this exercise, we assume that H, c and g are constants such that $0 < c \leq 1$ and $g > 0$. We want to find values of α, β, γ, δ, σ and ζ such that the function

$$f(z, v) = \alpha z^2 + \beta zv + \gamma v^2 + \delta z + \sigma v + \zeta$$

can be used as an invariant to prove that $z \leq H$.

1. We have seen that, to prove that $f(z, v) \geq 0$ is an invariant of the dynamics $\mathbf{e} = (\dot{z} = v, \dot{v} = -g)$, it is enough to prove that the Lie derivative $\mathcal{L}_{\mathbf{e}} f$ is the zero function. Compute this Lie derivative and deduce that, for this Lie derivative to be the zero function, it is enough to have :
$$\alpha = \beta = \sigma = 0 \quad \text{and} \quad \delta = 2\gamma g.$$
2. Now, we assume that f is of the form
$$f(z, v) = \gamma v^2 + 2\gamma gz + \zeta.$$
We have to prove that $f(z, v) \geq 0$ is preserved by the jump function of the **bouncing** event, that is :
$$f(z, v) \geq 0 \Rightarrow f(z, -cv) \geq 0.$$
Prove that this implication holds if $\gamma \leq 0$.
3. Next, we have to prove that $z \leq H$ is implied by the invariant, that is :
$$f(z, v) \geq 0 \Rightarrow z \leq H.$$
Prove that this implication holds if
$$\gamma < 0 \quad \text{and} \quad -\gamma v^2 \geq 2\gamma gH + \zeta \text{ for all } v.$$
Deduce that the latter inequality holds if $\zeta = -2\gamma gH$.
4. Summarising all the conditions so far, we have f of the form :
$$f(z, v) = \gamma v^2 + 2\gamma gz - 2\gamma gH$$
with $\gamma < 0$. To conclude, it is enough to prove that $f(z, v) \geq 0$ holds initially. Prove it.