Techniques de réécriture

TD n°4 : Confluence*

Exercise 1 :

Compute the critical pairs of the following rewrite systems. Which one are locally confluent ? convergent ?

1) $f(g(f(x))) \to x,\ f(g(x)) \to g(f(x))$
2) $0 + y \to y,\ x + 0 \to x,\ s(x) + y \to s(x + y),\ x + s(y) \to s(x + y)$
3) $f(x, x) \to a,\ f(x, g(x)) \to b$
4) $f(f(x, y), z) \to f(x, f(y, z)),\ f(x, 1) \to x$
5) $f(f(x, y), z) \to f(x, f(y, z)),\ f(1, x) \to x$
6) $f(x, f(y, z)) \to f(f(x, y), f(x, z)), f(f(x, y), z) \to f(f(x, z), f(y, z)), f(f(x, y), f(y, z)) \to y$

Exercise 2 :

Let $P = (\alpha_i, \beta_i)_{1\leq i\leq n}$ be an instance of PCP. Define $R(P) = \{A \to f(\alpha_i(\epsilon), \beta_i(\epsilon)), f(x, y) \to f(\alpha_i(x), \beta_i(y)), f(x, x) \to B, f(x, y) \to A\}$ on $\mathcal{F} = \{f(2), A(0), B(0), 0(1), 1(1), \epsilon(0)\}$.

1) Prove that P has a solution iff $A \to^* B$.
2) Deduce that confluence is undecidable.

Definition. We say that a rewrite system is orthogonal if it is left-linear and has no critical pair.

Definition. We say that a set of positions is parallel if any two distinct elements of this set are incomparable for the prefix order.
Let s be a term and $P = \{p_1, ..., p_n\}$ a parallel set of positions of s. Let t_p for all $p \in P$ be terms. We define $s[t_p]_{p\in P}$ by $s[t_{p_1}]_{p_1}...[t_{p_n}]_{p_n}$. Notice that the order is irrelevant because P is parallel.
If for each $p \in P$ and if we assume given a rule $l_p \to r_p \in R$ and a substitution σ_p such that $s_{|p} = l_p\sigma_p$, we write $s \rightrightarrows s[r_p\sigma_p]_{p\in P}$ and call it a parallel reduction step.
Notice that $\to \subseteq \rightrightarrows \subseteq \to^*$.

Exercise 3 :

Consider the combinatorial logic $CI = \{@(I, x) \to x, @(@(K, x), y) \to x, @(@(@(S, x), y), z) \to @(@(x, z), @(y, z))\}$.

1) Define a term Ω such that for all term t, $@(\Omega, t) \to^+ @(t, t)$. Deduce that CI does not terminate.
2) Prove the parallel moves lemma : if $l \to r \in R$ is a left-linear rule, if $l\sigma \rightrightarrows^P t$ and if all elements of P are below some variable position of l, then there is a substitution σ' such that $r\sigma \rightrightarrows r\sigma'$ and $t \to r\sigma'$.
3) Deduce that if R is orthogonal then $\rightrightarrows$ has the diamond property i.e. if $s_1 \leftleftarrows t \rightrightarrows s_2$ then there exists w such that $s_1 \rightrightarrows w \leftleftarrows s_2$.

*taken from *Term Rewriting and All That* and *Advanced Topics in Term Rewriting*

4) Conclude that if R is orthogonal, then R is confluent.
5) Deduce that CI is confluent.