Techniques de réécriture

TD n°1 : Well quasi-orderings

Those exercises are based on :
— S. Schmitz and P. Schnoebelen, *Algorithmic aspects of WQO theory*, lecture notes
— J. Goubault-Larrecq, *On Noetherian spaces*

Definition. Some definitions from ordering theory :
- a pre-ordering is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary reflexive and transitive relation on A
- given a pre-ordering $(A, \leq)$, we note $<$ the relation defined by :

$$x < y \Leftrightarrow x \leq y \wedge y \not\leq x$$

- we say that a pre-ordering $(A, \leq)$ is total if for every x, $y \in A$, $x \leq y$ or $y \leq x$
- we say that a subset U of A is upward-closed if for every $x \in U$ and every $y \in A$ such that $x \leq y$ then $y \in U$
- we say that $x \in A$ is minimal iff there are no $y \in A$ such that $y < x$
- we say that two elements x, $y \in A$ are equivalent iff $x \leq y$ and $y \leq x$
- given a subset U of A, we note $\uparrow U$ the upward-closure of U i.e. $\{x \in A \mid \exists y \in U,\, y \leq x\}$
- a linearization of $(A, \leq)$ is a total pre-ordering $(A, \sqsubseteq)$ such that :

$$x \leq y \Rightarrow x \sqsubseteq y$$

$$x < y \Rightarrow x \sqsubset y$$

Definition (wqo1)**.** A wqo is a pre-ordering $(A, \leq)$ such that every infinite sequence $(x_i)_{i \in \mathbb{N}}$ over A has an increasing pair i.e. there are $i < j \in \mathbb{N}$ such that $x_i \leq x_j$.

Definition (wqo2)**.** A wqo is a pre-ordering $(A, \leq)$ such that every infinite sequence $(x_i)_{i \in \mathbb{N}}$ over A has an infinite increasing subsequence i.e. there are $i_0 < i_1 < \ldots < i_k < \ldots \in \mathbb{N}$ such that $x_{i_0} \leq x_{i_1} \leq \ldots \leq x_{i_k} \leq \ldots$

Definition (wqo3)**.** A wqo is a pre-ordering $(A, \leq)$ such that :

well-founded : there are no infinite strictly decreasing sequences i.e. no sequences $x_0 > x_1 > \ldots > x_k > \ldots$ in A

no infinite antichains : there are no infinite subsets of A of mutually incomparable elements i.e. such that $x \not\leq y$ and $y \not\leq x$

Definition (wqo4)**.** A wqo is a pre-ordering $(A, \leq)$ such that any increasing sequence $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_k \subseteq \ldots$ of upward-closed subsets of A stabilizes i.e. there is $p \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $U_{p+i} = U_p$.

Exercise 1 :

Prove the equivalence of the four definitions of wqo.
hint : you can use (prove it) this particular case of the infinite Ramsey's theorem : let X be a countable set, X_2 be the set of all subsets of X of cardinal 2 and Σ be a finite set. Then, for every function $f : X_2 \longrightarrow \Sigma$, there is a infinite subset Y of X such that the restriction $f_Y : Y_2 \longrightarrow \Sigma$ is constant.

Solution:

wqo2 ⇒ wqo1 : OK

wqo1 ⇒ wqo3 : OK

proof of the hint : Let us construct by induction on $n \in \mathbb{N}$ sets Y_n and Z_n such that Y_n is infinite and $Z_n \cap Y_n = \varnothing$:

— $Y_0 = X$ and $Z_0 = \varnothing$

— Assume Y_n and Z_n constructed. Take $a_{n+1} \in Y_n$. Then, we have a partition of $Y_n \subsetneq \{a_{n+1}\}$ with the sets $Y_{n+1,c} = \{a \in Y_n \subsetneq \{a_{n+1}\} \mid f(\{a, a_{n+1}\}) = c\}$ for $c \in \Sigma$. As Σ is finite and $Y_n \subsetneq \{a_{n+1}\}$ is infinite, there is at least a $c \in \Sigma$ such that $Y_{n+1,c}$ is infinite. Take, $Y_{n+1} = Y_{n+1,c}$ for this c and $Z_{n+1} = Z_n \cup \{a_{n+1}\}$.

By construction, Z_n is of cardinal n and for all $k < n, m$, $f(\{a_k, a_n\}) = f(\{a_k, a_m\} = c_k$. Then, if we call $Z = \bigcup_{n \in \mathbb{N}} Z_n$ which is countable, we have a partition of Z by the sets $Z_c = \{k \in \mathbb{N} \mid c_k = c\}$ for $c \in \Sigma$. As Σ is finite, there is a c such that Z_c is infinite. Take $Y = Z_c$ for this c.

wqo3 ⇒ wqo2 : Let S be a infinite sequence x_0, x_1, ..., x_n, ... over A. Define the function $f : S_2 \longrightarrow \{1, 2, 3\}$ such that for every $\{x_i, x_j\}$ with $i < j$:

— either $x_i \leq x_j$ then $f(\{x_i, x_j\}) = 1$

— either $x_i > x_j$ then $f(\{x_i, x_j\}) = 2$

— either x_i and x_j are incomparable then $f(\{x_i, x_j\}) = 3$

Then by the hint, there is a infinite set Y such that f is constant on Y_2 :

— either its value is 1, then it gives what we want

— either its value is 2, then it gives an infinite strictly decreasing sequence which contradicts well foundedness

— either its value is 3, then it gives an infinite anti chain which contradicts wqo3

wqo4 ⇒ wqo1 : Take an infinite sequence x_0, ..., x_n, ... over A and define $U_i = \uparrow \{x_0, ..., x_i\}$. Then $U_0 \subseteq U_1 \subseteq ....$ So there is $k \in \mathbb{N}$ such that $U_{k+1} = U_k$ i.e. $x_{k+1} \in \uparrow \{x_0, ..., x_k\}$ i.e. there $i \leq k$ such that $x_i \leq x_{k+1}$.

wqo1 ⇒ wqo4 : Assume that you have $U_0 \subsetneq U_1 \subsetneq ...$ a strictly increasing sequence of upward-closed subsets of A. For all i, take $x_i \in U_{i+1} \subsetneq U_i$. Then $(x_i)_{i \in \mathbb{N}^*}$ contradicts wqo1 because the U_i are upward-closed.

Exercise 2 :

Which ones are wqo ?

1) $\mathbb{N}, \leq$
2) $\mathbb{Z}, \leq$
3) $\mathbb{N}, \mid$ where $\mid$ is the divisibility relation
4) prefix ordering on a finite alphabet
5) lexicographic ordering on $\{1, ..., n\}$ i.e. $a_0...a_k <_{lex} b_0...b_m$ iff $a_0...a_k$ is a prefix of $b_0...b_m$ or there is $i \leq min\{n, m\}$ such that for all $0 \leq j < i$, $a_j = b_j$ and $a_i < b_i$
6) $\mathcal{P}(\mathbb{N}), \subseteq$
7) $\mathcal{P}(\mathbb{N}), \sqsubseteq$ where $U \sqsubseteq V$ iff for all $m \in V$, there is $n \in U$ such that $n \leq m$
8) $R = \{(a, b) \in \mathbb{N}^2 \mid a < b\}$ with $(a, b) \leq (a', b')$ iff $(a = a' \wedge b \leq b') \vee b < a'$

Solution:

1) Yes, it is total and well-founded.
2) No, $(-n)_{n \in \mathbb{N}}$ is strictly decreasing.
3) No, the set of prime numbers is an infinite antichain.
4) For $n = 1$, yes, it is the example 1. For $n > 2$, no, $(a^n b)_{n \in \mathbb{N}}$ is an infinite antichain.
5) For $n = 1$, yes, it is the example 1. For $n > 2$, no, $(1^n 2)_{n \in \mathbb{N}}$ is strictly decreasing.
6) No, $(\{n\})_{n \in \mathbb{N}}$ is an infinite antichain.
7) Yes, because $U \sqsubseteq V$ iff $minU \leq minV$.
8) Yes. First, you have to check it is a pre-ordering (do all the cases, this is not difficult). Then, it is well founded because for every x, the set $\{y \in R \mid y \leq x\}$ is finite. Let an antichain $((a_i, b_i))_{i \in I}$. By the first part of the definition of $\leq$, for all $i \neq j$, $a_i \neq b_i$.

Now assume that $I = \mathbb{N}$. Then, by the previous remark, there is a i such that $b_0 < a_i$ which contradicts the fact it is an antichain.

Exercise 3 :

1) Show that every element of a well-founded pre-ordering is larger than or equal to a minimal element.
2) Prove that a pre-ordering $(A, \leq)$ is a wqo iff every non-empty subset of A has at least one minimal element and at most a finite number of minimal elements up to equivalence.
3) Prove that any upward-closed subset of a wqo can be written as $\uparrow \{x_1, ..., x_n\}$ for some x_1, ..., x_n.

Solution:

1) Let $x \in A$ well founded. Assume that there are no y minimal such that $y \leq x$. We construct an infinite strictly decreasing sequence by induction :
   — $x_0 = x$
   — Assume that $x_i < x_{i-1} < ... < x_0 = x$ are constructed. By hypothesis, x_i is not minimal, then there is $x_{i+1} < x_i$.
2) $\Rightarrow$) As a wqo is well founded, then all subset of a wqo is well founded. If this subset is non-empty, then by 1), it has at least a minimal element. Assume that there is an infinite number of minimal elements up to equivalence i.e. you have a sequence $(x_i)_{i \in \mathbb{N}}$ with x_i minimal and for $i \neq j$, with x_i and x_j non equivalent. This means that either $x_i \not\leq x_j$ either $x_j \not\leq x_i$. As they are minimal, x_i and x_j are incomparable and then it gives an infinite antichain.
   $\Leftarrow$) **well founded :** An infinite strictly decreasing sequence gives non-empty subset which has no minimal element.
   no infinite antichain : An infinite antichain gives an subset which has an infinite number of incomparable elements which are minimal in this subset.
3) Let U be upward-closed. By 2), it has a finite number of minimal elements up to equivalence. Let x_1, ..., x_n be elements representing those equivalence class. Let us prove that $U = \uparrow \{x_1, ..., x_n\}$:
   $\subseteq$) U is well founded because A is, and so by 1), every element of U is greater than a minimal element and so than a x_i (because every minimal element is equivalent to a x_i).
   $\supseteq$) All the x_i belongs to U and U is upward-closed.

Exercise 4 :

1) Prove that a total pre-ordering is a wqo iff it is well-founded.
2) Prove that every pre-ordering has a linearization.
   hint : use Zorn's lemma
3) Prove that a pre-ordering is a wqo iff all its linearizations are well-founded.

Solution:

1) A total pre-ordering cannot have infinite anti chains.
2) Let $\Gamma = \{(A, \sqsubseteq) \mid \leq \subseteq \sqsubseteq \wedge < \subseteq \sqsubset \wedge \subseteq \text{ is a pre-ordering}\}$. It is an inductive set, so has a maximal element $(A, \prec)$. In particular, $\prec$ is a pre-ordering an to prove it is linearization of $(A, \leq)$, we have to show that it is total. Suppose it is not. So there are $x \neq y$ such that $x \not\preceq y$ and $y \not\preceq x$. Define $\preceq' = (\preceq \cup \{(x, y)\})^*$. Then $\preceq'$ is a pre-ordering such that $\leq \subseteq \preceq \subseteq \preceq'$. To have a contradiction, it remains to prove that $\prec \subseteq \prec'$. Assume $\alpha \prec \beta$. So $\alpha \preceq \beta$ and $\beta \not\preceq \alpha$. Then $\alpha \preceq' \beta$. Assume $\beta \preceq' \alpha$, this means that $\beta \preceq x$ and $y \preceq \alpha$. Thus, $y \preceq \alpha \preceq \beta \preceq x$ which is absburd.
3) $\Rightarrow$) Let $\preceq$ be a linearization of $\leq$. Assume that you have $x_1 \succ x_2 \succ ...$, then $x_1 \not\leq x_2 \not\leq ...$ which contradicts wqo1.
   $\Leftarrow$) **well founded :** a strictly decreasing sequence for $\leq$ is also a strictly decreasing sequence for any linearization (and there is at least one by 2))

no infinite antichain : assume that you have an infinite antichain $(x_i)_{i\in\mathbb{N}}$. Let $\leq' = (\leq \cup \{(x_j, x_i) \mid i < j\})^*$. This is a pre-ordering which satisfies :

- $\leq \subseteq \leq'$
- $< \subseteq <'$: if $x \leq y$ and $y \not\leq x$ then $x \leq' y$. Assume that $y \leq' x$ then there are $i < j$ such that $y \leq x_j$ and $x_i \leq x$ and so $x_i \leq x_j$ which is absurd.

So any linearization of $\leq'$ is a linearization of $\leq$. But $\leq'$ is not well founded so one linearization obtained in 2) is not too which is absurd.

Definition. A bit of general topology :

- a topology on a set X is a set (of open subsets) $\mathcal{O}(X) \subseteq \mathcal{P}(X)$ such that $\varnothing$ and $X \in \mathcal{O}(X)$ and $\mathcal{O}(X)$ is closed by unions and finite intersections
- a topological space is a set with a topology on it
- given a pre-ordering $(A, \leq)$, its Alexandrov's topology is the one such that $\mathcal{O}(X)$ is the set of upward-closed subsets of A
- a subset K of X is said to be compact if for every family $(U_i)_{i\in I} \subseteq \mathcal{O}(X)$ such that $K \subseteq \bigcup\limits_{i\in I} U_i$ there a finite subset J of I such that $K \subseteq \bigcup\limits_{i\in J} U_i$
- we say that a topological space is Noetherian iff every open is compact

Exercise 5 :

1) Prove that the Alexandrov's topology is a topology.
2) Prove that a pre-ordering $(A, \leq)$ is a wqo iff A with its Alexandrov's topology is Noetherian.

Solution:

1) Easy.
2) $\Rightarrow$) Let U be an upward-closed and $(U_i)_{i\in I}$ be a family of upward-closed sets such that $U \subseteq \bigcup\limits_{i\in I} U_i$. By exercise 3, $U = \uparrow \{x_1, ..., x_n\}$ for some x_1, ..., x_n. Let i_j such that $x_j \in U_{i_j}$ and $J = \{i_j \mid j \in \{1, ..., n\}\}$. As the U_i are upward-closed, $U \subseteq \bigcup\limits_{i\in J} U_i$.

   $\Leftarrow$) Given a sequence $U_1 \subseteq U_2 \subseteq ...$ of upward-closed sets. Let $U = \bigcup\limits_{i\in\mathbb{N}} U_i$, which is an upward-closed. By hypothesis, there are $i_1 < ... < i_k$ such that $U = \bigcup\limits_{i\in\{i_1,...,i_k\}} U_i$ and so $U = U_{i_k}$.