Techniques de réécriture

TD n°1 : Termination & interpretations

Exercise 1 :

Are those rewrite systems terminating ?

- $\{f(a) \to f(b) \ ; \ g(b) \to g(a)\}$
- $\{f(f(x)) \to f(g(f(x)))\}$
- $\{f(g(x)) \to g(f(x))\}$
- $\{f(g(x)) \to g(g(f(f(x))))\}$

We recall that a polynomial interpretation on integers is the following data :
— a subset A of $\mathbb{N}$;
— for every symbol f of arity n, a polynomial $P_f \in \mathbb{Z}[X_1, ..., X_n]$
satisfying that for every symbol f of arity n :
— for every a_1, ..., $a_n \in A$, $P_f(a_1, ..., a_n) \in A$
— for every a_1, ..., $a_i > a'_i$, ..., $a_n \in A$, $P_f(a_1, ..., a_i, ..., a_n) > P_f(a_1, ..., a'_i, ..., a_n)$.
Then $(A, (P_f)_f, >)$ is a well-founded monotone algebra.

Exercise 2 :

Prove the termination of the following rewrite systems using the given polynomial interpretation on integers :

1. $\{x+0 \to x \ ; \ x+s(y) \to s(x+y) \ ; \ d(0) \to 0 \ ; \ d(s(x)) \to s(s(d(x))) \ ; \ c(0) \to 0 \ ; \ c(s(x)) \to c(x) + s(d(x))\}$ on $\mathbb{N} \setminus \{0, 1\}$ with $P_+(X, Y) = X + 2Y$, $P_s(X) = X + 1$, $P_d(X) = 3X$, $P_c = X^3$ and $P_0 = 2$.
2. $\{f(f(x, y), z) \to f(x, f(y, z)) \ ; \ f(y, f(x, z)) \to f(x, x)\}$ on $\mathbb{N} \setminus \{0, 1, 2\}$ with $P_f = X^2 + XY$.

Exercise 3 :

Prove the termination of the following rewrite system by finding a polynomial interpretation on integers :

$$\{x \times (y + z) \to (x \times y) + (x \times z) \ ; \ (x + y) + z \to x + (y + z)\}$$

Exercise 4 :

Let R be a rewrite system whose termination can be proved using a polynomial interpretation on integers. Let $A \subseteq \mathbb{N}$ be its domain and for every f in the alphabet F, P_f the interpretation of f. Take $a \in A \setminus \{0\}$.

1. Define $\pi_a : T(F, X) \longrightarrow A \setminus \{0\}$ the function which maps every variable x to a and every term of the form $f(t_1, \ldots, t_n)$ to $P_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. Prove that $\pi_a(t)$ is greater or equal to the length of every reduction starting from t.
2. Show that there exists d and k integers such that $a \le d$ and for every $f \in F$ of arity n and every a_1, ..., $a_n \in A \setminus \{0\}$, $P_f(a_1, \ldots, a_n) \le d \prod_{i=1}^{n} a_i^k$.
3. Fix $c \ge k + \log_2(d)$. Prove that $\pi_a(t) \le 2^{2^{c|t|}}$.

4. In this question, consider any finite rewrite system and $f \in F$. Prove that there exists an integer k such that if $s \to t$ then $|t|_f \leq k(|s|_f + 1)$, where $|.|_f$ is the number of f.
5. Deduce that $\{a(0, y) \to s(y)\ ;\ a(s(x), 0) \to a(x, s(0))\ ;\ a(s(x), s(y)) \to a(x, a(s(x), y))\}$ cannot be proved terminating using a polynomial interpretation on integers.
   hint : you may use the fact that the Ackermann's function grows faster than any primitive recursive function.

A polynomial interpretation on real numbers is the following data :
— a subset A of $\mathbb{R}_+$;
— a positive real number δ ;
— for every symbol f of arity n, a polynomial $P_f \in \mathbb{R}[X_1, ..., X_n]$
satisfying that for every symbol f of arity n :
— for every a_1, ..., $a_n \in A$, $P_f(a_1, ..., a_n) \in A$
— for every a_1, ..., $a_i >_\delta a'_i$, ..., $a_n \in A$, $P_f(a_1, ..., a_i, ..., a_n) >_\delta P_f(a_1, ..., a'_i, ..., a_n)$, where $x >_\delta y$ iff $x > y + \delta$.
Then $(A, (P_f)_f, >_\delta)$ is a well-founded monotone algebra.

Exercise 5 :
Define the following two rewrite systems :

$$R_1 = \{f(g(x)) \to g(g(f(x)))\ ;\ g(s(x)) \to s(s(g(x)))\ ;\ g(x) \to h(x, x)\ ;\ s(x) \to h(x, 0)\ ;\ s(x) \to h(x, 0)\}$$

$$R_2 = \{k(k(k(x))) \to h(k(x), k(x))\ ;\ s(h(k(x), k(x))) \to k(k(k(x)))\}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers : $\delta = 1$, $P_0 = 0$, $P_s = X + 4$, $P_f = X^2$, $P_g = 3X + 5$, $P_h = X + Y$ and $P_k = \sqrt{2}X + 1$.
2. Prove that any polynomial interpretation on integers proving the termination of R_1 is of the form $P_s = X + s_0$, $P_h = X + Y + h_0$, $P_g = g_1 X + g_0$ with s_0, h_0, $g_0 \geq 1$, $g_1 \geq 2$ and P_f of degree at least 2.
   hint : look at the dominant terms of the polynomials computed from the rewrite rules. For example, from the second rule of R_1, you should be able to prove that P_s is of degree 1.
3. Deduce that you cannot prove the termination of $R_1 \cup R_2$ using a polynomial interpretation on integers.

A matrix interpretation on integers is the following data :
— a positive integer d ;
— for every symbol of arity n, n matrices $M_{f,1}$, ..., $M_{f,n} \in \mathbb{N}^{d \times d}$;
— for every symbol of arity n, a vector $V_f \in \mathbb{N}^d$;
— a non-empty set $I \subseteq \{1, ..., d\}$
satisfying that for every symbol f of arity n, the map

$$L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \quad (X_1, ..., X_n) \mapsto \sum M_{f,i}.X_i + V_f$$

is monotonic with respect to $>_I$ where $X >_I Y$ iff for every $i \in \{1, ..., d\}$, $X[i] \geq Y[i]$ and there is $j \in I$ such that $X[j] > Y[j]$.
Then $(\mathbb{N}^d, (L_f)_f, >_I)$ is a well-founded monotone algebra.

Exercise 6 :

Consider the following rewrite system :

$$R = \{f(a) \to f(g(a)) \ ; \ g(b) \to g(f(b))\}$$

1. Prove that the termination of R cannot be proved by a polynomial interpretation on integers.
2. Prove the termination of R using the following matrix interpretation with $>_{\{1,2\}}$:

$$L_f(X) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} X \qquad L_g(X) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} X \qquad L_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad L_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3. Why does it fail if we take $>_{\{1\}}$ instead ? Is there another matrix interpretation that works with this ordering ?

Exercise 7 :

Prove that the following rewrite system :

$$\{f(g(x)) \to f(a(g(g(f(x))), g(g(f(x))))) \ ; \ h(h(x)) \to c(h(x)) \ ; \ a(x,x) \to h(x) \ ;$$

$$c(x) \to x \ ; \ f(x) \to x \ ; \ g(x) \to x\}$$

using the following matrix interpretation with $>_{\{1\}}$:

$$L_a(X,Y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad L_c(X) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$L_f(X) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad L_g(X) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} X + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad L_h(X) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Why does it fail with $>_{\{1,2\}}$? Is there another matrix interpretation that works with this ordering ?

Exercise 8 :

Prove that termination is undecidable :

(data) R finite rewrite system.

(question) Does R terminate ?

hint : reduce the Post correspondence problem. From an instance of the PCP, construct a rewrite system that terminates iff the PCP has no solutions.