Techniques de réécriture

TD n°2 : KBO & confluence*

Definition. We recall that a strict order $>$ on $T(\mathcal{F}, \mathcal{X})$ is a simplification order if :

compatibility : $\forall s_1 > s_2$, $t_1, ..., t_{i-1}, t_{i+1}, ..., t_n$ and $f \in \mathcal{F}$,

$$f(t_1, ..., t_{i-1}, s_1, t_{i+1}, ..., t_n) > f(t_1, ..., t_{i-1}, s_2, t_{i+1}, ..., t_n)$$

substitution closure : $\forall s_1 > s_2$, σ substitution, $s_1\sigma > s_2\sigma$

subterm property : $\forall t$, $p \in Pos(t) \setminus \{\epsilon\}$, $t > t_{|p}$

Definition. Fix $>$ a strict order on $\mathcal{F}$. A weight function is a function $w : \mathcal{F} \cup \mathcal{X} \longrightarrow \mathbb{R}_+$.
We say that w is admissible w.r.t. $>$ if :

— $\exists w_0 \in \mathbb{R}_+^*$ such that $\forall x \in \mathcal{X}$, $w(x) = w_0$ and for all constant c, $w(c) \geq w_0$

— if $f \in \mathcal{F}$ is of arity 1 and $w(f) = 0$ then f is maximal for $>$

We extend a weight function on $T(\mathcal{F}, \mathcal{X})$ by : $w(t) = \sum\limits_{x \in \mathcal{X}} w(x). \mid t \mid_x + \sum\limits_{f \in \mathcal{F}} w(f). \mid t \mid_f$.

Definition. Fix a strict order $>$ on $\mathcal{F}$ and a weight function w admissible w.r.t. $>$. The Knuth-Bendix order $<_{kbo}$ induced on $T(\mathcal{F}, \mathcal{X})$ is defined by $s <_{kbo} t$ if $\forall x \in \mathcal{X}$, $\mid s \mid_x \leq \mid t \mid_x$ and if one of the following holds :

(KBO1) $w(s) < w(t)$

(KBO2) $w(s) = w(t)$ and one of the following holds

(KBO2a) $\exists f \in \mathcal{F}$, $x \in \mathcal{X}$ and $k > 0$ such that $s = x$ and $t = f^k(x)$

(KBO2b) $\exists f > g$ such that $s = g(s_1, ..., s_m)$ and $t = f(t_1, ..., t_n)$

(KBO2c) $\exists f \in \mathcal{F}$ and $i \leq n$ such that $s = f(s_1, ..., s_n)$, $t = f(t_1, ..., t_n)$, $s_1 = t_1$, ..., $s_{i-1} = t_{i-1}$ and $s_i <_{kbo} t_1$

You can admit that $<_{kbo}$ is a strict order (prove it if you have time).

Exercise 1 :

Prove that $<_{kbo}$ is simplification order.

Exercise 2 :

Which rewrite systems from TD 1 are terminating using KBO ?

Exercise 3 :

Compute the critical pairs of the following rewrite systems. Which one are locally confluent ? convergent ?

- $f(g(f(x))) \to x$, $f(g(x)) \to g(f(x))$
- $0 + y \to y$, $x + 0 \to x$, $s(x) + y \to s(x + y)$, $x + s(y) \to s(x + y)$
- $f(x, x) \to a$, $f(x, g(x)) \to b$
- $f(f(x, y), z) \to f(x, f(y, z))$, $f(x, 1) \to x$
- $f(f(x, y), z) \to f(x, f(y, z))$, $f(1, x) \to x$
- $f(x, f(y, z)) \to f(f(x, y), f(x, z))$, $f(f(x, y), z) \to f(f(x, z), f(y, z))$, $f(f(x, y), f(y, z)) \to y$

*taken from *Term Rewriting and All That*

Definition. We say that a rewrite system is orthogonal if it is left-linear and has no critical pair.

Definition. We say that a set of positions is parallel if any two distinct elements of this set are incomparable for the prefix order.
Let s be a term and $P = \{p_1, ..., p_n\}$ a parallel set of positions of s. Let t_p for all $p \in P$ be terms. We define $s[t_p]_{p\in P}$ by $s[t_{p_1}]_{p_1}...[t_{p_n}]_{p_n}$. Notice that the order is irrelevant because P is parallel.
If for each $p \in P$, we are given a rule $l_p \to r_p \in R$ and a substitution σ_p such that $s_{|p} = l_p\sigma_p$, we write $s \rightrightarrows s[r_p\sigma_p]_{p\in P}$ and call it a parallel reduction step.
Notice that $\to \subseteq \rightrightarrows \subseteq \to^*$.

Exercise 4 :

1) Prove that if R has no critical pair and if there exists two rules $l_1 \to r_1$ and $l_2 \to r_2$ and substitutions σ_1 and σ_2 such that $l_1\sigma_1 = l_2\sigma_2$ then $r_1\sigma_1 = r_2\sigma_2$.
2) Prove that if R is orthogonal then $\rightrightarrows$ has the diamond property i.e. if $s_1 \leftleftarrows t \rightrightarrows s_2$ then there exists w such that $s_1 \rightrightarrows w \leftleftarrows s_2$.
3) Conclude that if R is orthogonal, then R is confluent.
4) Deduce that the combinatorial logic $\{@(I, x) \to x, @(@(K, x), y) \to x, @(@(@(S, x), y), z) \to @(@(x, z), @(y, z))\}$ is confluent (the alphabet is $\{I(0), S(0), K(0), @(2)\}$).