Techniques de réécriture

TD n°1 : Proofs of termination *

Exercise 1 :

Are those rewrite systems terminating ?

- $\{f(a) \to f(b), g(b) \to g(a)\}$
- $\{f(f(x)) \to f(g(f(x)))\}$
- $\{f(g(x)) \to g(f(x))\}$
- $\{f(g(g(f(x)))) \to g(f(f(g(x))))\}$
- $\{f(g(x)) \to g(g(f(f(x))))\}$
- $\{f(g(g(x))) \to g(f(f(g(x))))\}$

Exercise 2 :

1) Prove that $R_1 = \{or(x,y) \to x, or(x,y) \to y\}$ terminates.
2) Prove that $R_2 = \{f(a,b,x) \to f(x,x,x)\}$ terminates.
3) What can you say about $R_1 \cup R_2$?

Exercise 3 :

Prove that those functions terminate using rewrite systems :

let rec Ack = function
  $(0, x)$ -> $x + 1$
  | $(x, 0)$ -> $\text{Ack}(x-1, 1)$
  | (x, y) -> $\text{Ack}(x-1, \text{Ack}(x, y-1))$; ;

let rec f = function
  $(0, x)$ -> 1
  | $(x, 0)$ -> 1
  | (x, y) -> $f(x-1, x-1) + 2 * f(x-1, y-1) + f(y-1, y-1)$; ;

Exercise 4 :

Heracles vs Lernaean Hydra. The Hydra is a term on the alphabet $\mathcal{F} = \{n(*), h(0)\}$. When Heracles cuts a head, an arbitrary number of heads grows back this way :

$$n(x_1, ..., x_p, n(y_1, ..., y_q, h), z_1, ..., z_r) \to n(x_1, ..., x_p, n(y_1, ..., y_q), h, ..., h, z_1, ..., z_r)$$

Heracles wins when there is no head to cut i.e. the term representing the Hydra is irreducible. Prove that Heracles always wins i.e. this rewrite system is terminating.

Exercise 5 :

We say that a rewrite system $\mathcal{R}$:

- goes around in circles when $\exists t,\ t \to^+_{\mathcal{R}} t$
- loops when $\exists t,\ t',\ t \to^+_{\mathcal{R}} t'$ and t is a sub-term of t'
- is self-imbricated when $\exists t,\ t',\ t \to^+_{\mathcal{R}} t'$ and $t \trianglelefteq t'$
- weakly terminates when $\forall t,\ \exists t',\ t \to^+_{\mathcal{R}} t'$ and t' irreducible

1) Prove the following implications :
   goes around in circles $\Rightarrow$ loops $\Rightarrow$ does not terminate $\Rightarrow$ is self-imbricated
2) Give a counter-example for each inverse implication.
3) Give an example of a rewrite system which weakly terminates but does not terminate.

Exercise 6 :

For which values of n, k, m and p does the rewrite system $\{f^n(g^k(x)) \to g^m(f^p(x))\}$ terminate ?

*taken from *Les termes en logique et en programmation*