Techniques de réécriture

TD n°2 : Termination by extensions, RPO and KBO

Exercise 1 :

Prove that for every n, $k > 0$, $p \geq 0$, the rewrite system with one rule $f^n(g^k(x)) \rightarrow g^k(f^p(x))$ terminates.

Exercise 2 :

1. Prove that those functions terminate :
   let rec Ack = function
   $(0, x)$ -> $x + 1$
   | $(x, 0)$ -> $\text{Ack}(x-1, 1)$
   | (x, y) -> $\text{Ack}(x-1, \text{Ack}(x, y-1))$; ;
   let rec f = function
   $(0, x)$ -> 1
   | $(x, 0)$ -> 1
   | (x, y) -> $f(x-1, x-1) + 2 * f(x-1, y-1) + f(y-1, y-1)$; ;
2. What does that prove about the following rewrite systems :
$$\{a(0, x) \rightarrow s(x)\ ;\ a(s(x), 0) \rightarrow a(x, s(0))\ ;\ a(s(x), s(y)) \rightarrow a(x, a(s(x), y))\}$$
$$\{f(0, x) \rightarrow s(0)\ ;\ f(x, 0) \rightarrow s(0)\ ;\ f(s(x), s(y)) \rightarrow g(f(x, x), h(f(x, y)), f(y, y))\}$$
3. Prove that those rewrite systems terminate by using a RPO. Idem by using lexicographic and multiset extensions.

Exercise 3 :

Heracles vs Lernaean Hydra. The Hydra is a term on the alphabet $\mathcal{F} = \{n(*), h(0)\}$. When Heracles cuts a head, an arbitrary number of heads grows back as follows :

$$n(x_1, ..., x_p, n(y_1, ..., y_q, h), z_1, ..., z_r) \rightarrow n(x_1, ..., x_p, n(y_1, ..., y_q), h, ..., h, z_1, ..., z_r)$$

Heracles wins when there is no head to cut i.e. the term representing the Hydra is irreducible. Prove that Heracles always wins i.e. this rewrite system is terminating.

Fix $>$ a strict order on $\mathcal{F}$. A weight function is a function $w : \mathcal{F} \cup \mathcal{X} \longrightarrow \mathbb{R}_+$.
We say that w is admissible w.r.t. $>$ if :
- $\exists w_0 \in \mathbb{R}_+^*$ such that $\forall x \in \mathcal{X}$, $w(x) = w_0$ and for all constant c, $w(c) \geq w_0$.
- if $f \in \mathcal{F}$ is of arity 1 and $w(f) = 0$ then f is maximal for $>$.

We extend a weight function on $T(\mathcal{F}, \mathcal{X})$ by : $w(t) = \sum\limits_{x \in \mathcal{X}} w(x).\mid t \mid_x + \sum\limits_{f \in \mathcal{F}} w(f).\mid t \mid_f$.

Fix a weight function w admissible w.r.t. $>$. The Knuth-Bendix order $<_{kbo}$ induced on $T(\mathcal{F}, \mathcal{X})$ is defined by $s <_{kbo} t$ if $\forall x \in \mathcal{X}$, $\mid s \mid_x \leq \mid t \mid_x$ and if one of the following holds :
(KBO1) $w(s) < w(t)$
(KBO2) $w(s) = w(t)$ and one of the following holds
(KBO2a) $\exists f \in \mathcal{F}$, $x \in \mathcal{X}$ and $k > 0$ such that $s = x$ and $t = f^k(x)$
(KBO2b) $\exists f > g$ such that $s = g(s_1, ..., s_m)$ and $t = f(t_1, ..., t_n)$
(KBO2c) $\exists f \in \mathcal{F}$ and $i \leq n$ such that $s = f(s_1, ..., s_n)$, $t = f(t_1, ..., t_n)$, $s_1 = t_1$, ..., $s_{i-1} = t_{i-1}$ and $s_i <_{kbo} t_1$

Exercise 4 :

1. Assume that f is of arity 1, $w(f) = 0$ and that there is g with $f \not> g$. Prove that $<_{kbo}$ does not satisfy the subterm property.
2. Assume that $w(s) = w(t)$ and that t is a strict subterm of s. Prove that there f of arity 1 with $w(f) = 0$ and $k \geq 1$ such that $s = f^k(t)$.
3. Prove that $<_{kbo}$ is a strict order. *Spoil : transitivity is hard.*
4. Prove that $<_{kbo}$ is a rewrite order, i.e., closed under context and substitution.
5. Prove that $<_{kbo}$ has the subterm property, and so is a simplification order.

Exercise 5 :

Show that you cannot prove the termination of the rewrite system defining the Ackermann's function using a KBO.

Exercise 6 :

Show that you cannot prove the termination of :

$$\{f(f(x,y),z) \to f(x,f(y,z)) \ ; \ f(y,f(x,z)) \to f(x,x)\}$$

using a RPO or a KBO.

Exercise 7 :

1. Prove the termination of :

   $$\{s(x) + (y+z) \to x + (s(s(y)) + z) \ ; \ s(x) + (y + (z+w)) \to x + (z + (y+w))\}$$

   using a KBO.
2. Show that you cannot prove its termination using a RPO or a polynomial interpretation on integers.