Techniques de réécriture

TD n°5 : Equational theories & completion [*]

Definition. Fix an alphabet $\mathcal{F}$ and a countably infinite set of variables $\mathcal{X}$. An identity is a pair $(s,t) \in T(\mathcal{F},\mathcal{X}) \times T(\mathcal{F},\mathcal{X})$. Fix a set E of identities. We define $\approx_E$, the induced equality, by the following rules :

$$\frac{(s,t) \in E}{s \approx_E t} \qquad \frac{}{t \approx_E t} \qquad \frac{s \approx_E t}{t \approx_E s} \qquad \frac{s \approx_E t \quad t \approx_E u}{s \approx_E u}$$

$$\frac{s \approx_E t}{s\sigma \approx_E t\sigma} \qquad \frac{s_1 \approx_E t_1 \ \cdots \ s_n \approx_E t_n}{f(s_1, ..., s_n) \approx_E f(t_1, ..., t_n)}$$

Definition. We are interested in the two following problems :
(Word Problem for E) (with E fixed) :
Instance : two terms s and t.
Question : $s \approx_E t$?

(Word Problem) :
Instance : a finite set E of identities and two terms s and t.
Question : $s \approx_E t$?

Exercise 1 :

1) Do you know a finite set of identities for which the word problem is undecidable ?
2) What can you say about **(Word Problem)** ?

Definition. We say that two sets of identities E and E' are equivalent if $\approx_E = \approx_{E'}$.

Exercise 2 :

What is the behavior of the basic completion procedure on the following set of identities (with the suitable reduction order) :

- $\{(x*(y+z), (x*y)+(x*z)), ((u+v)*w, (u*w)+(v*w))\}$ and $>$ the lpo with $* > +$
- $\{(x+0, x), (x+s(y), s(x+y))\}$ and $>$ the kbo with $s > +$ and weight 1 for all variables and symbols
- $\{(f(g(f(x))), x)\}$
- $\{(f(g(f(x))), f(g(x)))\}$
- $\{((x*y)*(y*z), y)\}$ with any simplification order

Exercise 3 :

1) Apply the Huet's completion procedure on $\{(f(x,y), h(x,c)), (f(x,y), h(c,y)), (h(x,c), c)\}$ and the lpo with $f > h > c$.
2) Prove that $R = \{f(x,y) \to c, h(c,y) \to c, h(x,c) \to c\}$ is equivalent to the set of identities of question 1.
3) Notice that R terminates. Prove this termination using the lpo of question 1. Deduce that R is convergent.

[*] taken from *Term Rewriting and All That* and *Proofs by induction in equational theories with constructors*

Algorithm 1 Basic completion procedure

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

if there exists $(s,t) \in E$ such that $s \neq t$, $s \not> t$ and $t \not> s$ **then**
  terminates with output FAIL
else
  $i := 0$
  $R_0 := \{(l,r) \mid (l,r) \in E \cup E^{-1} \wedge l > r\}$
end if
repeat
  $R_{i+1} := R_i$
  for all $(s,t) \in CP(R_i)$ **do**
    Reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
    if $\tilde{s} \neq \tilde{t} \wedge \tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**
      terminates with output FAIL
    end if
    if $\tilde{s} > \tilde{t}$ **then**
      $R_{i+1} := R_{i+1} \cup \{(\tilde{s},\tilde{t})\}$
    end if
    if $\tilde{t} > \tilde{s}$ **then**
      $R_{i+1} := R_{i+1} \cup \{(\tilde{t},\tilde{s})\}$
    end if
  end for
  $i := i + 1$
until $R_i = R_{i+1}$
return R_i

Exercise 4 :

1) Prove that the set of identities

$$\{(@(nil,x),x),$$
$$(@(cons(x,y),z), cons(x,@(y,z))),$$
$$(rev(nil),nil),(rev(cons(x,y)),$$
$$@(rev(y),cons(x,nil)))\}$$

on the ranked alphabet $\{nil(0), rev(1), cons(2), @(2)\}$ can be oriented to give a convergent TRS. Let R this TRS.

2) Prove that the associativity A of $@$:

$$@(@(x,y),z) = @(x,@(y,z))$$

is not a consequence of R.

3) Prove that we can complete (A,R).

4) Prove that the idempotence I of rev :

$$rev(rev(x)) = x$$

is not a consequence of R.

5) Prove that we can complete (I,R).

6) Prove that Huet's completion fails to complete $(\{rev(x) = @(x,x)\}, R)$.

Algorithm 2 Huet's completion procedure

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

$R_0 := \varnothing$; $E_0 := E$; $i := 0$
while $E_i \neq \varnothing$ or there is an unmarked rule in R_i **do**
  while $E_i \neq \varnothing$ **do**
    Choose an identity $(s,t) \in E$
    Reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
    if $\tilde{s} = \tilde{t}$ **then**
      $R_{i+1} := R_i$; $E_{i+1} := E_i \setminus \{(s,t)\}$; $i := i+1$
    else
      if $\tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**
        terminates with output FAIL
      else
        let l and r such that $\{l,r\} = \{\tilde{s},\tilde{t}\}$ and $l > r$
        $R_{i+1} := \{(g,\tilde{d}) \mid (g,d) \in R_i \wedge g$ cannot be reduced with $l \to r \wedge \tilde{d}$ is a $R_i \cup \{(l,r)\}$-normal form of $d\} \cup \{(l,r)\}$
        (l,r) is not marked and $(g,\tilde{d})$ is marked in R_{i+1} iff (g,d) is in R_i
        $E_{i+1} := (E_i \setminus \{(s,t)\}) \cup \{(g',d) \mid (g,d) \in R_i \wedge g$ can be reduced to g' with $l \to r\}$
        $i := i+1$
      end if
    end if
  end while
  if there is an unmarked rule in R_i **then**
    let (l,r) be such a rule
    $R_{i+1} := R_i$
    $E_{i+1} := \{(s,t) \mid (s,t)$ is a critical pair of (l,r) with itself or with a marked rule in $R_i\}$
    $i := i+1$
    Mark (l,r)
  end if
end while
return R_i