Techniques de réécriture

TD n°3 : Termination using dependency

Exercise 1 :

We consider the following rewriting system R :

$$\{m(x,0) \to x\ ;\ m(s(x),s(y)) \to m(x,y)\ ;\ q(0,s(y)) \to 0\ ;\ q(s(x),s(y)) \to s(q(m(x,y),s(y)))\ ;$$

$$p(0,y) \to y\ ;\ p(s(x),y) \to s(p(x,y))\ ;\ m(m(x,y),z) \to m(x,p(y,z))\}$$

The goal of this exercise is to prove its termination using the theory of dependency pairs.

1. Show that you cannot prove termination of R using a RPO or a KBO.
2. What are the defined symbols ?
3. Compute the marked dependency pairs.
4. Which pairs are connectable ? Draw the dependency graph approximation.
5. Are those pairs connected in the dependency graph ?
6. What are the inequalities that are enough to consider ? In particular, why the following inequalities :

$$q^{\#}(s(x),s(y)) \geq m^{\#}(x,y) \qquad m^{\#}(m(x,y),z) \geq p^{\#}(y,z)$$

   can be ignored ?

7. Find a weakly monotonic polynomial interpretation on integers satisfying those inequalities.

Exercise 2 :

An argument filtering TRS (AFTRS for short) on F is a rewrite system A on $F \sqcup F'$ for some F' such that the rules of A are of the form :

— either $f(x_1,...,x_n) \to g(y_1,...,y_k)$ with $f \in F$, $g \in F'$, the x_i are pairwise different, $y_j \in \{x_1,...,x_n\}$ and are pairwise different ;

— either $f(x_1,...,x_n) \to x_i$ with $f \in F$ and the x_j pairwise different.

Moreover, for every symbol $f \in F$, there is at most one rule of this form.

1. Prove that an AFTRS always terminates and is confluent.

Fix A an AFTRS and note t_A the normal form of t with respect to A. An inequality problem is a pair $IN_{\geq}$, $IN_{>}$ of relations on terms.

2. Prove that if there is $\geq$, a well founded weakly monotonic quasi-ordering on terms on $F \sqcup F'$, closed under substitution such that :
   — for every $(s,t) \in IN_{\geq}$, $s_A \geq t_A$;
   — for every $(s,t) \in IN_{>}$, $s_A > t_A$.
   then there is such quasi-order $\geq'$ on F satisfying $IN_{\geq} \subseteq \geq'$ and $IN_{>} \subseteq >'$.
3. Find a AFTRS for which you can apply question 2. on the set of inequalities obtained in exercice 1, question 6 with a RPO.

Exercise 3 :

We consider the following rewriting system R :

$$\{f(1) \to 1 \ ; \ f(h(x,y)) \to h(s(x), f(y)) \ ; \ g(0,0) \to h(0,1) \ ; g(s(x),0) \to 1 \ ;$$

$$g(s(x), s(y)) \to f(g(x,y)) \ ; \ g(0, s(y)) \to h(0, g(s(0), s(y)))\}$$

1. Show that you cannot prove the termination of R using a RPO. Which rule is problematic ? Intuitively, why many orders will fail to prove termination ?
2. Compute the dependency pairs.
3. Compute the dependency graph (approximation).

Assume that the defined symbols are all of arity ≥ 1. Let S be a rewrite system and fix C a cycle of its dependency graph. A simple projection for C is a mapping π from every marked symbol $f^{\#}$ of C of arity n to a number in $\{1, ..., n\}$. Note then $\pi(C)$ the set

$$\{(t_{\pi(f^{\#})}, s_{\pi(g^{\#})}) \mid (t,s) \in C\}.$$

4. Prove that if every cycle of the dependency graph of S has a simple projection π such that $\pi(C) \subseteq \trianglerighteq$ and $\pi(C) \cap \triangleright \neq \varnothing$, where $\trianglerighteq$ is the sub-term ordering, then S terminates.
5. Prove termination of R using question 4.