Techniques de réécriture

TD n°2 : Well quasi-orderings and ordering extensions

Those exercises are based on :
- S. Schmitz and P. Schnoebelen, *Algorithmic aspects of WQO theory* (lecture notes)
- F. Baader and T. Nipkov, *Term rewriting and all that* (book)
- J. Goubault-Larrecq, *On Noetherian spaces* (paper)

Definition. Some definitions from ordering theory :
- a quasi-ordering is a pair $(A, \leq)$ where A is a set and $\leq$ is a binary reflexive and transitive relation on A
- given a quasi-ordering $(A, \leq)$, we note $<$ the relation defined by :

$$x < y \Leftrightarrow x \leq y \wedge y \not\leq x$$

- we say that a quasi-ordering $(A, \leq)$ is total if for every x, $y \in A$, $x \leq y$ or $y \leq x$
- we say that a subset U of A is upward-closed if for every $x \in U$ and every $y \in A$ such that $x \leq y$ then $y \in U$
- we say that $x \in A$ is minimal iff there are no $y \in A$ such that $y < x$
- we say that two elements x, $y \in A$ are equivalent iff $x \leq y$ and $y \leq x$ (we may note $x \approx y$)
- given a subset U of A, we note $\uparrow U$ the upward-closure of U i.e. $\{x \in A \mid \exists y \in U, y \leq x\}$
- a linearization of $(A, \leq)$ is a total quasi-ordering $(A, \sqsubseteq)$ such that :

$$x \leq y \Rightarrow x \sqsubseteq y$$

$$x < y \Rightarrow x \sqsubset y$$

Definition (wqo1)**.** A wqo is a quasi-ordering $(A, \leq)$ such that every infinite sequence $(x_i)_{i \in \mathbb{N}}$ over A has an increasing pair i.e. there are $i < j \in \mathbb{N}$ such that $x_i \leq x_j$.

Definition (wqo2)**.** A wqo is a quasi-ordering $(A, \leq)$ such that every infinite sequence $(x_i)_{i \in \mathbb{N}}$ over A has an infinite increasing subsequence i.e. there are $i_0 < i_1 < \ldots < i_k < \ldots \in \mathbb{N}$ such that $x_{i_0} \leq x_{i_1} \leq \ldots \leq x_{i_k} \leq \ldots$

Definition (wqo3)**.** A wqo is a quasi-ordering $(A, \leq)$ such that :

well-founded : there are no infinite strictly decreasing sequences i.e. no sequences $x_0 > x_1 > \ldots > x_k > \ldots$ in A

no infinite antichains : there are no infinite subsets of A of mutually incomparable elements i.e. such that $x \not\leq y$ and $y \not\leq x$

Property (lecture)**.** *wqo1, 2 and 3 are equivalent.*

Exercise 1 :

Which ones are wqo ?
1) $\mathbb{N}, \leq$
2) $\mathbb{Z}, \leq$
3) $\mathbb{N}, \mid$ where $\mid$ is the divisibility relation
4) prefix ordering on a finite alphabet
5) lexicographic ordering on $\{1, ..., n\}$ i.e. $a_0...a_k <_{lex} b_0...b_m$ iff $a_0...a_k$ is a prefix of $b_0...b_m$ or there is $i \leq min\{n, m\}$ such that for all $0 \leq j < i$, $a_j = b_j$ and $a_i < b_i$

6) $\mathcal{P}(\mathbb{N}), \subseteq$
7) $\mathcal{P}(\mathbb{N}), \sqsubseteq$ where $U \sqsubseteq V$ iff for all $m \in V$, there is $n \in U$ such that $n \leq m$
8) $R = \{(a,b) \in \mathbb{N}^2 \mid a < b\}$ with $(a,b) \leq (a',b')$ iff $(a = a' \wedge b \leq b') \vee b < a'$

Solution:

1) Yes, it is total and well-founded.
2) No, $(-n)_{n\in\mathbb{N}}$ is strictly decreasing.
3) No, the set of prime numbers is an infinite antichain.
4) For $n = 1$, yes, it is the example 1. For $n > 2$, no, $(a^n b)_{n\in\mathbb{N}}$ is an infinite antichain.
5) For $n = 1$, yes, it is the example 1. For $n > 2$, no, $(1^n 2)_{n\in\mathbb{N}}$ is strictly decreasing.
6) No, $(\{n\})_{n\in\mathbb{N}}$ is an infinite antichain.
7) Yes, because $U \sqsubseteq V$ iff $minU \leq minV$.
8) Yes. First, you have to check it is a quasi-ordering (do all the cases, this is not difficult). Then, it is well founded because for every x, the set $\{y \in R \mid y \leq x\}$ is finite. Let an antichain $((a_i, b_i))_{i\in I}$. By the first part of the definition of $\leq$, for all $i \neq j$, $a_i \neq b_i$. Now assume that $I = \mathbb{N}$. Then, by the previous remark, there is a i such that $b_0 < a_i$ which contradicts the fact it is an antichain.

Definition (product extension)**.** Let $(D_1, \leq_1)$, ..., $(D_n, \leq_n)$ be non-empty quasi-orderings. Their product extension is the quasi ordering $(D_1 \times ... \times D_n, \leq_\times)$ with :

$$(x_1, ..., x_n) \leq_\times (y_1, ..., y_n) \iff \forall i,\ x_i \leq_i y_i$$

Exercise 2 :

Prove that $(D_1, \leq_1)$, ..., $(D_n, \leq_n)$ are well-founded (resp. a wqo) iff their product extension is well-founded (resp. a wqo).

Solution:

well-founded, $\Rightarrow$: Take a strictly decreasing sequence $(d_1^1, ..., d_1^n) > ... > (d_k^1, ..., d_k^n) >$ .... For $j \in \{1, ..., n\}$, let $\Gamma_j = \{k \in \mathbb{N} \mid d_k^j >_j d_{k+1}^j\}$. Then, by hypothesis, $\bigcup_{j=1}^{n} \Gamma_j = \mathbb{N}$.

So by the pigeonhole principle, there is a j such that Γ_j is infinite. Then $(d_k^j)_{k\in\Gamma_j}$ is a strictly decreasing sequence in D_j. Absurd.

wqo, $\Rightarrow$: We will prove wqo2. Let A be an infinite sequence $((d_k^1, ..., d_k^n))_{k\in\mathbb{N}}$. We construct an infinite subsequence of A by induction on j this way :

- $\Gamma_0 = A$
- Assume Γ_j constructed. Then $\pi_{j+1}(\Gamma_j)$ where π_i is the ith projection, is an infinite sequence in D_{j+1}. So, by wqo2, there is an infinite subsequence Γ_{j+1} of Γ_j such that $\pi_{j+1}(\Gamma_j)$ is increasing.

Then Γ_n is an infinite increasing subsequence of A.

well-founded, $\Leftarrow$: Assume that you have an infinite strictly decreasing sequence $x_0 >_i x_1 >_i ... >_i x_k >_i ...$ for some i. For $j \neq i$, take $y_j \in D_j \neq \varnothing$.
Then $((y_1, ..., y_{i-1}, x_k, y_{i+1}, ..., y_n))_{k\in\mathbb{N}}$ is an infinite strictly decreasing sequence in the product extension. Absurd.

wqo, $\Leftarrow$: Idem with antichains.

Definition (lexicographic extension)**.** Let $(D_1, \leq_1)$, ..., $(D_n, \leq_n)$ be non-empty quasi-orderings. Their lexicographic extension is the quasi ordering $(D_1 \times ... \times D_n, \leq_{lex})$ with :

$$(x_1, ..., x_n) \leq_{lex} (y_1, ..., y_n) \iff (\forall i,\ x_i \approx_i y_i) \vee (\exists j,\ (\forall i < j,\ x_i \approx_i y_i) \wedge x_j <_j y_j)$$

Exercise 3 :

Prove that $(D_1, \leq_1)$, ..., $(D_n, \leq_n)$ are well-founded (resp. a wqo) iff their lexicographic extension is well-founded (resp. a wqo).

Solution:

well-founded, $\Rightarrow$: We prove it by induction on n. For $n = 0$, OK. Assume it true for $n-1$. Take a strictly decreasing sequence $(d_1^1, ..., d_1^n) > ... > (d_k^1, ..., d_k^n) > ....$ Then, $(d_k^1)_{k\in\mathbb{N}}$ is decreasing. So, there is a i_1 such that for all $k \geq i_1$, $d_k^1 \approx_1 d_{i_1}^1$. Then, $((d_k^2, ..., d_k^n))_{k\geq i_1}$ is a strictly decreasing sequence. Absurd by induction hypothesis.

wqo, $\Rightarrow$: If two elements of the lexicographic extension are incomparable then one of their components are incomparable. Then by the pigeonhole principle, any infinite antichain gives rise two an infinite antichain in some component. Absurd.

$\Leftarrow$: Same as the product.

Definition (multiset extension)**.** Let $(D, \leq)$ be a partial-ordering. Its multiset extension is the quasi-ordering $(Mul(D), \sqsubseteq)$ with $Mul(D)$ is the set of finite multistep and $M \sqsubset N$ iff $\exists X, Y \in Mul(D)$:

— $\varnothing \neq X \subseteq N$
— $M = (N - X) + Y$
— $\forall y \in Y, \exists x \in X, x > y$

Exercise 4 :

1) Prove that $\sqsubset$ is irreflexive and transitive.
2) Prove that $(D, \leq)$ is well-founded iff its multiset extension is well-founded.

Solution:

1) **irreflexive** : If $M \sqsubset M$ then necessarily $X = Y \neq \varnothing$. Then the last condition will not be satisfied on a maximal element of Y (which exists because Y is finite).
transitive : Assume $M \sqsubset M'$ with $M = (M' - X) + Y$ and $M' \sqsubset M''$ with $M' = (M'' - X') + Y'$. Let $X'' = (X - Y') + X'$ and $Y'' = Y + (Y' - X)$. Then $M = (M'' - X'') + Y''$ and satisfy the conditions.
2) Assume given an infinite strictly decreasing sequence $M_0 \sqsupset M_1 \sqsupset ... \sqsupset M_k \sqsupset ...$ in $Mul(D)$. We construct a tree labelled in $D \sqcup \{\top, \bot\}$ like this :
— the root is labelled by $\top$
— for every occurrence of every element of M_0, construct a son of the root labelled by this element. This is the step 0
— assume constructed the steps $\leq i$. For every occurrence of every element of X_i with $M_{i+1} = (M_i - X_i) + Y_i$, choose a node which has no son labelled by this element (two different occurrences of the same element must correspond to different nodes). For each of those nodes, construct a son labelled by $\bot$. For every occurrence of every element $y \in Y_i$, choose one the previous nodes which is labelled by an element $x \in X_i$ such that $x > y$ and construct a son of this node labelled by y. This is the step $i + 1$.

To prove the correction of this construction, you can prove that at the step i, the multiset of labels of leaves which are in D is exactly M_i. This tree is infinite because at each step we had at least a node $\bot$, as X_i is non-empty. It is finitely branching because Y_i is finite. So, by the König lemma, it has an infinite branch, which gives an infinite strictly decreasing sequence in D. Absurd.

Exercise 5 :

Prove that a quasi-ordering is a wqo iff any increasing sequence $U_0 \subseteq U_1 \subseteq \dots \subseteq U_k \subseteq \dots$ of upward-closed subsets stabilizes i.e. there is $p \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $U_{p+i} = U_p$.

Solution:

wqo4 $\Rightarrow$ wqo1 : Take an infinite sequence $x_0, ..., x_n, ...$ over A and define $U_i = \uparrow \{x_0, ..., x_i\}$. Then $U_0 \subseteq U_1 \subseteq ....$ So there is $k \in \mathbb{N}$ such that $U_{k+1} = U_k$ i.e. $x_{k+1} \in \uparrow \{x_0, ..., x_k\}$ i.e. there $i \leq k$ such that $x_i \leq x_{k+1}$.

wqo1 $\Rightarrow$ wqo4 : Assume that you have $U_0 \subsetneq U_1 \subsetneq \dots$ a strictly increasing sequence of upward-closed subsets of A. For all i, take $x_i \in U_{i+1} \subseteq U_i$. Then $(x_i)_{i\in\mathbb{N}^*}$ contradicts wqo1 because the U_i are upward-closed.

Exercise 6 :

1) Show that every element of a well-founded quasi-ordering is larger than or equal to a minimal element.
2) Prove that a quasi-ordering $(A, \leq)$ is a wqo iff every non-empty subset of A has at least one minimal element and at most a finite number of minimal elements up to equivalence.
3) Prove that any upward-closed subset of a wqo can be written as $\uparrow \{x_1, ..., x_n\}$ for some x_1, ..., x_n.

Solution:

1) Let $x \in A$ well founded. Assume that there are no y minimal such that $y \leq x$. We construct an infinite strictly decreasing sequence by induction :
   — $x_0 = x$
   — Assume that $x_i < x_{i-1} < \dots < x_0 = x$ are constructed. By hypothesis, x_i is not minimal, then there is $x_{i+1} < x_i$.
2) $\Rightarrow$) As a wqo is well founded, then all subset of a wqo is well founded. If this subset is non-empty, then by 1), it has at least a minimal element. Assume that there is an infinite number of minimal elements up to equivalence i.e. you have a sequence $(x_i)_{i\in\mathbb{N}}$ with x_i minimal and for $i \neq j$, with x_i and x_j non equivalent. This means that either $x_i \not\leq x_j$ either $x_j \not\leq x_i$. As they are minimal, x_i and x_j are incomparable and then it gives an infinite antichain.

   $\Leftarrow$) **well founded :** An infinite strictly decreasing sequence gives non-empty subset which has no minimal element.

   no infinite antichain : An infinite antichain gives an subset which has an infinite number of incomparable elements which are minimal in this subset.
3) Let U be upward-closed. By 2), it has a finite number of minimal elements up to equivalence. Let x_1, ..., x_n be elements representing those equivalence class. Let us prove that $U = \uparrow \{x_1, ..., x_n\}$:
   $\subseteq$) U is well founded because A is, and so by 1), every element of U is greater than a minimal element and so than a x_i (because every minimal element is equivalent to a x_i).
   $\supseteq$) All the x_i belongs to U and U is upward-closed.

Exercise 7 :

1) Prove that a total quasi-ordering is a wqo iff it is well-founded.
2) Prove that every quasi-ordering has a linearization.
   hint : use Zorn's lemma
3) Prove that a quasi-ordering is a wqo iff all its linearizations are well-founded.

Solution:

1) A total quasi-ordering cannot have infinite anti chains.
2) Let $\Gamma = \{(A, \sqsubseteq) \mid \leq \subseteq \sqsubseteq \wedge < \subseteq \sqsubset \wedge \subseteq \text{ is a quasi-ordering}\}$. It is an inductive set, so has a maximal element $(A, \prec)$. In particular, $\prec$ is a quasi-ordering an to prove it is linearization of $(A, \leq)$, we have to show that it is total. Suppose it is not. So there are $x \neq y$ such that $x \not\preceq y$ and $y \not\preceq x$. Define $\preceq' = (\preceq \cup \{(x, y)\})^*$. Then $\preceq'$ is a quasi-ordering such that $\leq \subseteq \preceq \subseteq \preceq'$. To have a contradiction, it remains to prove that $\prec \subseteq \prec'$. Assume $\alpha \prec \beta$. So $\alpha \preceq \beta$ and $\beta \not\preceq \alpha$. Then $\alpha \preceq' \beta$. Assume $\beta \preceq' \alpha$, this means that $\beta \preceq x$ and $y \preceq \alpha$. Thus, $y \preceq \alpha \preceq \beta \preceq x$ which is absburd.
3) $\Rightarrow$) Let $\preceq$ be a linearization of $\leq$. Assume that you have $x_1 \succ x_2 \succ \dots$, then $x_1 \not\leq x_2 \not\leq \dots$ which contradicts wqo1.

   $\Leftarrow$) **well founded :** a strictly decreasing sequence for $\leq$ is also a strictly decreasing sequence for any linearization (and there is at least one by 2))

no infinite antichain : assume that you have an infinite antichain $(x_i)_{i\in\mathbb{N}}$. Let $\leq' = (\leq \cup \{(x_j, x_i) \mid i < j\})^*$. This is a quasi-ordering which satisfies :

— $\leq \subseteq \leq'$

— $< \subseteq <'$: if $x \leq y$ and $y \not\leq x$ then $x \leq' y$. Assume that $y \leq' x$ then there are $i < j$ such that $y \leq x_j$ and $x_i \leq x$ and so $x_i \leq x_j$ which is absurd.

So any linearization of $\leq'$ is a linearization of $\leq$. But $\leq'$ is not well founded so one linearization obtained in 2) is not too which is absurd.

Definition. A bit of general topology :

— a topology on a set X is a set (of open subsets) $\mathcal{O}(X) \subseteq \mathcal{P}(X)$ such that $\varnothing$ and $X \in \mathcal{O}(X)$ and $\mathcal{O}(X)$ is closed by unions and finite intersections

— a topological space is a set with a topology on it

— given a quasi-ordering $(A, \leq)$, its Alexandrov's topology is the one such that $\mathcal{O}(X)$ is the set of upward-closed subsets of A

— a subset K of X is said to be compact if for every family $(U_i)_{i\in I} \subseteq \mathcal{O}(X)$ such that $K \subseteq \bigcup\limits_{i\in I} U_i$ there a finite subset J of I such that $K \subseteq \bigcup\limits_{i\in J} U_i$

— we say that a topological space is Noetherian iff every open is compact

Exercise 8 :

1) Prove that the Alexandrov's topology is a topology.
2) Prove that a quasi-ordering $(A, \leq)$ is a wqo iff A with its Alexandrov's topology is Noetherian.

Solution:

1) Easy.
2) $\Rightarrow$) Let U be an upward-closed and $(U_i)_{i\in I}$ be a family of upward-closed sets such that $U \subseteq \bigcup\limits_{i\in I} U_i$. By exercise 3, $U = \uparrow \{x_1, ..., x_n\}$ for some x_1, ..., x_n. Let i_j such that $x_j \in U_{i_j}$ and $J = \{i_j \mid j \in \{1, ..., n\}\}$. As the U_i are upward-closed, $U \subseteq \bigcup\limits_{i\in J} U_i$.

$\Leftarrow$) Given a sequence $U_1 \subseteq U_2 \subseteq ...$ of upward-closed sets. Let $U = \bigcup\limits_{i\in\mathbb{N}} U_i$, which is an upward-closed. By hypothesis, there are $i_1 < ... < i_k$ such that $U = \bigcup\limits_{i\in\{i_1,...,i_k\}} U_i$ and so $U = U_{i_k}$.