Techniques de réécriture

TD n°3 : Equational theories & completion

Definition. Fix an alphabet $\mathcal{F}$ and a countably infinite set of variables $\mathcal{X}$. An identity is a pair $(s,t) \in T(\mathcal{F},\mathcal{X}) \times T(\mathcal{F},\mathcal{X})$. Fix a set E of identities. We define $\approx_E$, the induced equality, by the following rules :

$$\frac{(s,t) \in E}{s \approx_E t} \qquad \frac{}{t \approx_E t} \qquad \frac{s \approx_E t}{t \approx_E s} \qquad \frac{s \approx_E t \quad t \approx_E u}{s \approx_E u}$$

$$\frac{s \approx_E t}{s\sigma \approx_E t\sigma} \qquad \frac{s_1 \approx_E t_1 \quad \cdots \quad s_n \approx_E t_n}{f(s_1, ..., s_n) \approx_E f(t_1, ..., t_n)}$$

Definition. We are interested in the two following problems :
(Word Problem for E) (with E fixed) :
Instance : two terms s and t.
Question : $s \approx_E t$?

(Word Problem) :
Instance : a finite set E of identities and two terms s and t.
Question : $s \approx_E t$?

Exercise 1 :
1) Do you know a finite set of identities for which the word problem is undecidable ?
2) What can you say about **(Word Problem)** ?

Exercise 2 :
1) Given a set of identities E, prove that $R = E \cup E^{-1}$ is confluent.
2) Prove that the **(Confluence Problem)** :
   Instance : a finite rewrite system R.
   Question : Is R confluent ?
   is undecidable.

Definition. We say that two sets of identities E and E' are equivalent if $\approx_E = \approx_{E'}$.

Exercise 3 :
What is the behavior of the basic completion procedure on the following set of identities (with the suitable reduction order) :
- $\{(x * (y + z), (x * y) + (x * z)), ((u + v) * w, (u * w) + (v * w))\}$ and $>$ the lpo with $* > +$
- $\{(x + 0, x), (x + s(y), s(x + y))\}$ and $>$ the kbo with $s > +$ and weight 1 for all variables and symbols
- $\{(f(g(f(x))), x)\}$
- $\{(f(g(f(x))), f(g(x)))\}$
- $\{((x * y) * (y * z), y)\}$ with any simplification order

Algorithm 1 Basic completion procedure

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

if there exists $(s,t) \in E$ such that $s \neq t$, $s \not> t$ and $t \not> s$ **then**
    terminates with output FAIL
else
    $i := 0$
    $R_0 := \{(l,r) \mid (l,r) \in E \cup E^{-1} \wedge l > r\}$
end if
repeat
    $R_{i+1} := R_i$
    for all $(s,t) \in CP(R_i)$ **do**
        Reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
        if $\tilde{s} \neq \tilde{t} \wedge \tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**
            terminates with output FAIL
        end if
        if $\tilde{s} > \tilde{t}$ **then**
            $R_{i+1} := R_{i+1} \cup \{(\tilde{s}, \tilde{t})\}$
        end if
        if $\tilde{t} > \tilde{s}$ **then**
            $R_{i+1} := R_{i+1} \cup \{(\tilde{t}, \tilde{s})\}$
        end if
    end for
    $i := i + 1$
until $R_i = R_{i+1}$
return R_i

Exercise 4 :

You can admit the correction of the procedure.

1) Prove that the Huet's completion procedure always successfully terminates on finite sets of ground identities and $>$ a reduction order that is total on ground terms.
2) Deduce that **(Word Problem)** is decidable for finite sets of ground identities.

Exercise 5 :

1) Apply the Huet's completion procedure on $\{(f(x,y), h(x,c)), (f(x,y), h(c,y)), (h(x,c), c)\}$ and the lpo with $f > h > c$.
2) Prove that $R = \{f(x,y) \to c, h(c,y) \to c, h(x,c) \to c\}$ is equivalent to the set of identities of question 1.
3) Notice that R terminates. Prove this termination using the lpo of question 1. Deduce that R is convergent.

Exercise 6 :

1) Prove that both completion procedures may fail on

$$\{(h(x,y), f(x)), (h(x,y), f(y)), (g(x,y), h(x,y)), (g(x,y), a)\}$$

and the lpo with $g > h > f > a$.
2) How can we avoid the failure ?

Algorithm 2 Huet's completion procedure

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

$R_0 := \varnothing$; $E_0 := E$; $i := 0$
while $E_i \neq \varnothing$ or there is an unmarked rule in R_i **do**
  while $E_i \neq \varnothing$ **do**
    Choose an identity $(s, t) \in E$
    Reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
    if $\tilde{s} = \tilde{t}$ **then**
      $R_{i+1} := R_i$; $E_{i+1} := E_i \setminus \{(s, t)\}$; $i := i + 1$
    else
      if $\tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**
        terminates with output FAIL
      else
        let l and r such that $\{l, r\} = \{\tilde{s}, \tilde{t}\}$ and $l > r$
        $R_{i+1} := \{(g, \tilde{d}) \mid (g, d) \in R_i \wedge g$ cannot be reduced with $l \to r \wedge \tilde{d}$ is a $R_i \cup \{(l, r)\}$-normal form of $d\} \cup \{(l, r)\}$
        (l, r) is not marked and $(g, \tilde{d})$ is marked in R_{i+1} iff (g, d) is in R_i
        $E_{i+1} := (E_i \setminus \{(s, t)\}) \cup \{(g', d) \mid (g, d) \in R_i \wedge g$ can be reduced to g' with $l \to r\}$
        $i := i + 1$
      end if
    end if
  end while
  if there is an unmarked rule in R_i **then**
    let (l, r) be such a rule
    $R_{i+1} := R_i$
    $E_{i+1} := \{(s, t) \mid (s, t)$ is a critical pair of (l, r) with itself or with a marked rule in $R_i\}$
    $i := i + 1$
    Mark (l, r)
  end if
end while
return R_i