# Techniques de réécriture

## TD n°4 : Confluence & completion

**Exercise 1 :**

Compute the critical pairs of the following rewrite systems. Which one are locally confluent ?

1. $f(g(f(x))) \to x,\ f(g(x)) \to g(f(x))$
2. $0 + y \to y,\ x + 0 \to x,\ s(x) + y \to s(x + y),\ x + s(y) \to s(x + y)$
3. $f(x, x) \to a,\ f(x, g(x)) \to b$
4. $f(f(x, y), z) \to f(x, f(y, z)),\ f(x, 1) \to x$

**Exercise 2 :**

Let $P = (\alpha_i, \beta_i)_{1 \leq i \leq n}$ be an instance of PCP. Define $R(P) = \{A \to f(\alpha_i(\epsilon), \beta_i(\epsilon)), f(x, y) \to f(\alpha_i(x), \beta_i(y)), f(x, x) \to B, f(x, y) \to A\}$ on $\mathcal{F} = \{f(2), A(0), B(0), 0(1), 1(1), \epsilon(0)\}$.

1. Prove that $P$ has a solution iff $A \to^* B$.
2. Deduce that confluence is undecidable.

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**Algorithm 1** Basic completion procedure

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**Require:** A finite set $E$ of identities and a reduction order $>$
**Ensure:** A finite convergent rewrite system $R$ equivalent to $E$ if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

**if** there exists $(s, t) \in E$ such that $s \neq t$, $s \not> t$ and $t \not> s$ **then**
    terminates with output FAIL
**else**
    $i := 0$
    $R_0 := \{(l, r) \mid (l, r) \in E \cup E^{-1} \wedge l > r\}$
**end if**
**repeat**
    $R_{i+1} := R_i$
    **for all** $(s, t) \in CP(R_i)$ **do**
        Reduce $s$ and $t$ to some $R_i$-normal forms $\tilde{s}$ and $\tilde{t}$
        **if** $\tilde{s} \neq \tilde{t} \wedge \tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**
            terminates with output FAIL
        **end if**
        **if** $\tilde{s} > \tilde{t}$ **then**
            $R_{i+1} := R_{i+1} \cup \{(\tilde{s}, \tilde{t})\}$
        **end if**
        **if** $\tilde{t} > \tilde{s}$ **then**
            $R_{i+1} := R_{i+1} \cup \{(\tilde{t}, \tilde{s})\}$
        **end if**
    **end for**
    $i := i + 1$
**until** $R_i = R_{i+1}$
**return** $R_i$

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**Exercise 3 :**

We are considering this basic completion procedure.

1. Prove that this procedure is correct by showing it consists in a strategy for applying some rules from the completion procedure seen in the course.
2. Which rules are not used ?
3. What can you say about $\bigcup_{i \in \mathbb{N}} R_i$ if the procedure does not terminate ?

**Exercise 4 :**

Apply the basic completion procedure on the following set of identities, with the suitable reduction order :

1. $\{(x * (y + z), (x * y) + (x * z)), ((u + v) * w, (u * w) + (v * w))\}$ and $>$ the LPO with $* > +$.
2. $\{(x + 0, x), (x + s(y), s(x + y))\}$ and $>$ the KBO with $s > +$ and weight 1 for all variables and symbols.
3. $\{(f(g(f(x))), x)\}$ and the LPO with $f > g$.

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**Algorithm 2** Huet's completion procedure

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**Require:** A finite set $E$ of identities and a reduction order $>$

**Ensure:** A finite convergent rewrite system $R$ equivalent to $E$ if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

$R_0 := \varnothing$ ; $E_0 := E$ ; $i := 0$<br>
**while** $E_i \neq \varnothing$ or there is an unmarked rule in $R_i$ **do**<br>
  **while** $E_i \neq \varnothing$ **do**<br>
    Choose an identity $(s, t) \in E$<br>
    Reduce $s$ and $t$ to some $R_i$-normal forms $\tilde{s}$ and $\tilde{t}$<br>
    **if** $\tilde{s} = \tilde{t}$ **then**<br>
      $R_{i+1} := R_i$ ; $E_{i+1} := E_i \setminus \{(s, t)\}$ ; $i := i + 1$<br>
    **else**<br>
      **if** $\tilde{s} \not> \tilde{t} \wedge \tilde{s} \not> \tilde{t}$ **then**<br>
        terminates with output FAIL<br>
      **else**<br>
        let $l$ and $r$ such that $\{l, r\} = \{\tilde{s}, \tilde{t}\}$ and $l > r$<br>
        $R_{i+1} := \{(g, \tilde{d}) \mid (g, d) \in R_i \wedge g$ cannot be reduced with $l \to r \wedge \tilde{d}$ is a $R_i \cup \{(l, r)\}$-normal form of $d\} \cup \{(l, r)\}$<br>
        $(l, r)$ is not marked and $(g, \tilde{d})$ is marked in $R_{i+1}$ iff $(g, d)$ is in $R_i$<br>
        $E_{i+1} := (E_i \setminus \{(s, t)\}) \cup \{(g', d) \mid (g, d) \in R_i \wedge g$ can be reduced to $g'$ with $l \to r\}$<br>
        $i := i + 1$<br>
      **end if**<br>
    **end if**<br>
  **end while**<br>
  **if** there is an unmarked rule in $R_i$ **then**<br>
    let $(l, r)$ be such a rule<br>
    $R_{i+1} := R_i$<br>
    $E_{i+1} := \{(s, t) \mid (s, t)$ is a critical pair of $(l, r)$ with itself or with a marked rule in $R_i\}$<br>
    $i := i + 1$<br>
    Mark $(l, r)$<br>
  **end if**<br>
**end while**<br>
**return** $R_i$

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**Exercise 5 :**

We are now considering Huet's completion procedure.

1. Do the same study as exercise 4.

2. Prove that the set of identities
$$\{(@(nil, x), x),$$
$$(@(cons(x, y), z), cons(x, @(y, z))),$$
$$(rev(nil), nil), (rev(cons(x, y)),$$
$$@(rev(y), cons(x, nil)))\}$$
can be oriented to give a convergent TRS. Let $R$ this TRS.
3. Prove that the associativity $A$ of @, $@(@(x, y), z) = @(x, @(y, z))$ is not a consequence of $R$.
4. How would you prove associativity of concatenation of lists ?
5. Prove that you can complete $(A, R)$. You can use Huet's completion procedure.
6. Prove that the idempotence $I$ of $rev$, $rev(rev(x)) = x$ is not a consequence of $R$.
7. Prove that you can complete $(I, R)$.
8. Prove that Huet's completion fails to complete $(\{rev(x) = @(x, x)\}, R)$.