Partial Higher Dimensional Automata

Session “Young Mathematicians’ Challenge”

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Another aspect of CPS: Concurrency

Concurrent system:
A system with different agents accomplishing their tasks simultaneously, while communicating, competing on resources, ...

Examples:
- OS
- Computer (multi-core)
- Network
- CPS
- ...
Many models of concurrency

- Mutual-exclusion model (Dijkstra, 1965)
- Petri nets (Petri, 1962)
- Process algebra
  - Communicating Sequential Processes (Hoare, 1976)
  - Calculus of Communicating Systems (Milner, 1980)
  - π-calculus (Milner, Parrow, Walker, 1992)
- Parallel Random-Access Machine (Fortune, Wyllie, Goldshlager, 1974)
- Actor model (Hewitt, Bishop, Steiger, 1973)
- Bulk Synchronous Parallel (Valiant, 1990)
- Tuple spaces, Linda (Galernter, Carriero, 1986)
- Simple Concurrent Object Oriented Programming (Meyer, 1993)
- Reo Coordination Language (Arbab, 2004)
- ...

Interleaving concurrency...

\[ X := 0 \parallel Y := 1 \approx (X := 0 ; Y := 1) + (Y := 1 ; X := 0) \]

Two actions in parallel \( \approx \)
doing them sequentially, in any order, produces the same result
... vs true concurrency

Action refinement: “X := 0” and “Y := 1” are not atomic!
Many other executions between the two interleaved ones

==> Abstraction: there is a continuity of executions in-between
A geometric model of true concurrency

Higher Dimensional Automata
Precubical sets

A precubical set is:
- a collection of sets \((X_n)_{n \in \mathbb{N}}\),
- a collection of functions \((\partial^\alpha_{i,n} : X_n \rightarrow X_{n-1})_{n > 0, 1 \leq i \leq n, \alpha \in \{0,1\}}\)

satisfying for \(i > j\),
\[
\partial^\beta_{j,n} \circ \partial^\alpha_{i,n+1} = \partial^\alpha_{i-1,n} \circ \partial^\alpha_{j,n+1}
\]

Directed graph:
- \(X_0\) = set of vertices,
- \(X_1\) = set of edges,
- \(X_{n>1}\) = \(\emptyset\),
- \(\partial^0_{1,1}\) = source function,
- \(\partial^1_{1,1}\) = target function,
- equations are trivial.
A Higher Dimensional Automata is:

- a precubical set \((X, \partial)\),
- an initial state \(i_0 \in X_0\),
- a labelling function \(\lambda : X_1 \rightarrow \Sigma\)

satisfying for every \(c \in X_2\):

\[
\lambda(\partial_{i,2}^1(c)) = \lambda(\partial_{i,2}^0(c))
\]
HDA, categorically

The category of HDA is equivalent to the following double slice category of the category of presheaves over the cube category:

$$1/[\square^{op}, \text{Set}]/\Sigma$$
Allowing partiality

Category theory wins
Extending HDA?

How would you model: “Actions $a$ and $b$ can be done at the same time, but $a$ must start before $b$.”? 

States where $b$ has started but not $a$. Those must be removed.
In an elegant way

The category of partial HDA is the following double slice category of the category of lax functors over the cube category:

$$1/Lax(\square^{op}, pSet)/\Sigma$$

In short: replace some functions (corresponding to the face maps) by partial functions
In an ugly way

A **partial precubical set** is:
- a collection of sets \((X_n)_{n \in \mathbb{N}}\),
- a collection of *partial* functions \((\partial^{\alpha_1,\ldots,\alpha_k}_{i_1<\ldots<i_k,n} : X_n \to X_{n-k})_{n>0, 1 \leq k \leq n, \alpha_i \in \{0,1\}}\)

satisfying for \(i > j\),

\[
\partial^{\beta_1,\ldots,\beta_n}_{j_1<\ldots<j_n} \circ \partial^{\alpha_1,\ldots,\alpha_m}_{i_1<\ldots<i_m} \subseteq \partial^{\gamma_1,\ldots,\gamma_p}_{k_1<\ldots<k_p}
\]

A **partial Higher Dimensional Automata** is:
- a **partial** precubical set \((X, \partial)\),
- an initial state \(i_0 \in X_0\),
- a collection of labelling functions \((\lambda_n : X_n \to \Sigma^n)_{n \in \mathbb{N}}\)

satisfying for every \(c \in X_n\):

\[
\lambda_{n-1}(\partial^1_{i,n}(c)) = \lambda_{n-1}(\partial^0_{i,n}(c))
\]
So... now what?

- We have a geometric model of true concurrency.
  \[ HDA, pHDA \rightarrow \text{Top, dTop} \]

- We can use tools from mathematics (algebraic topology) to study those models.

- This gave rise to a new mathematical field, the “directed algebraic topology”

Come to see my poster to see what this is, and what we can do with algebraic topology in true concurrency!