Partial Higher Dimensional Automata

Session “Young Mathematicians’ Challenge”

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Another aspect of CPS: Concurrency

<u>**Concurrent system:**</u>

A system with different agents accomplishing their tasks simultaneously, while communicating, competing on resources, …

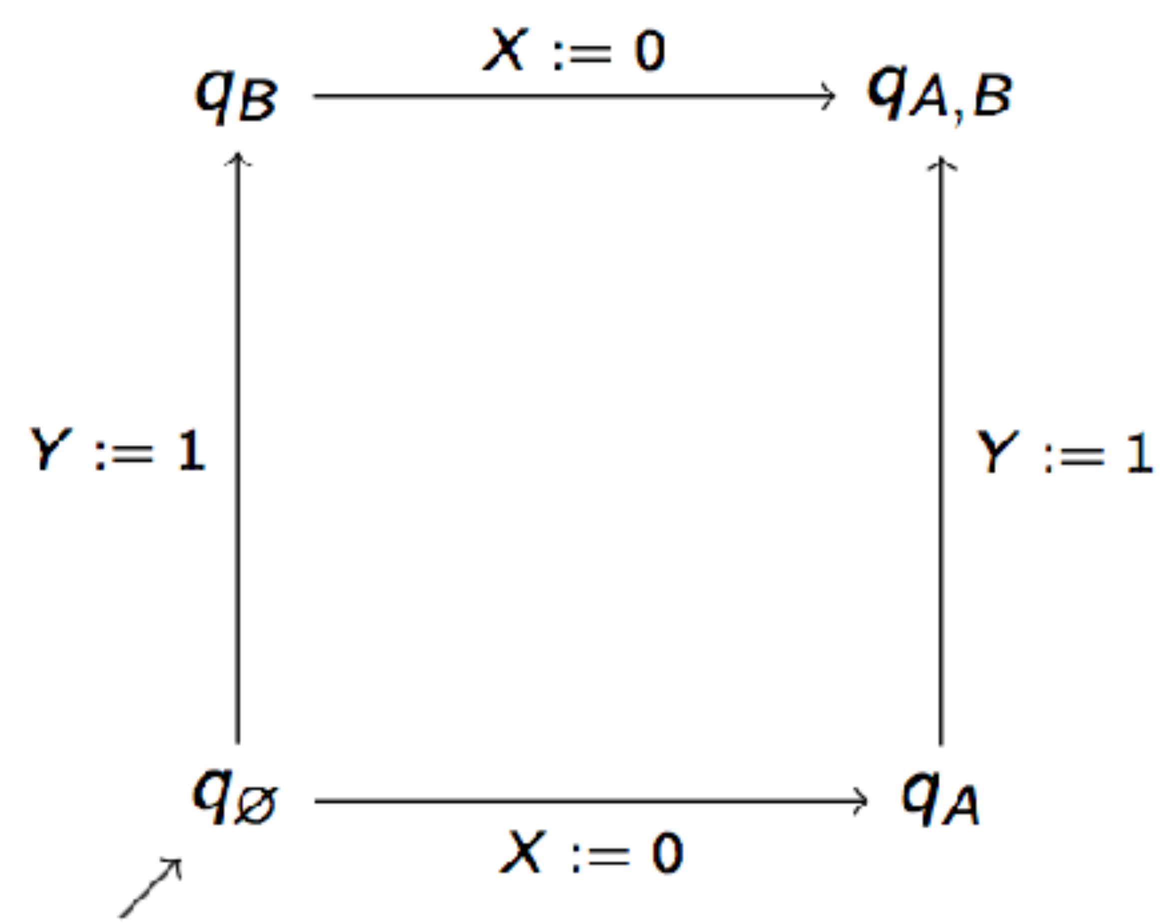
Examples:

- **OS**
- **Computer (multi-core)**
- **Network**
- **CPS**
- **…**

Many models of concurrency

- Mutual-exclusion model (Dijkstra, 1965)
- Petri nets (Petri, 1962)
- Process algebra
    - Communicating Sequential Processes (Hoare, 1976)
    - Calculus of Communicating Systems (Milner, 1980)
    - π-calculus (Milner, Parrow, Walker, 1992)
- Parallel Random-Access Machine (Fortune, Wyllie, Goldshlager, 1974)
- Actor model (Hewitt, Bishop, Steiger, 1973)
- Bulk Synchronous Parallel (Valiant, 1990)
- Tuple spaces, Linda (Galernter, Carriero, 1986)
- Simple Concurrent Object Oriented Programming (Meyer, 1993)
- Reo Coordination Language (Arbab, 2004)
- …

Interleaving concurrency…

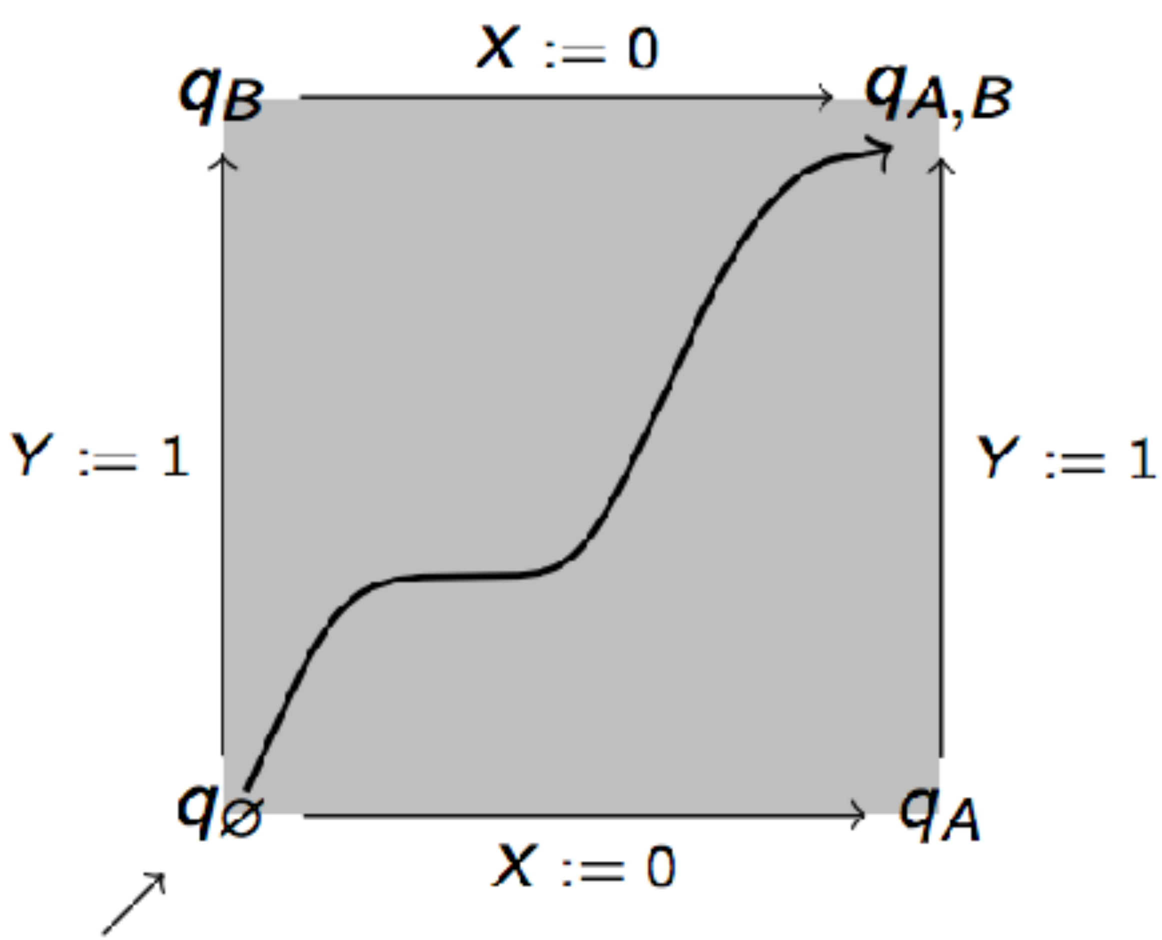


X := 0 || Y := 1 ≈ (X := 0 ; Y := 1) + (Y := 1 ; X := 0)

Two actions in parallel ≈

doing them sequentially, in any order, produces the same result

… vs true concurrency



Action refinement: “X := 0” and “Y := 1” are not atomic!
Many other executions between the two interleaved ones
==> Abstraction: there is a continuity of executions in-between

A geometric model of true concurrency

Higher Dimensional Automata

Precubical sets

A ***precubical set*** is:

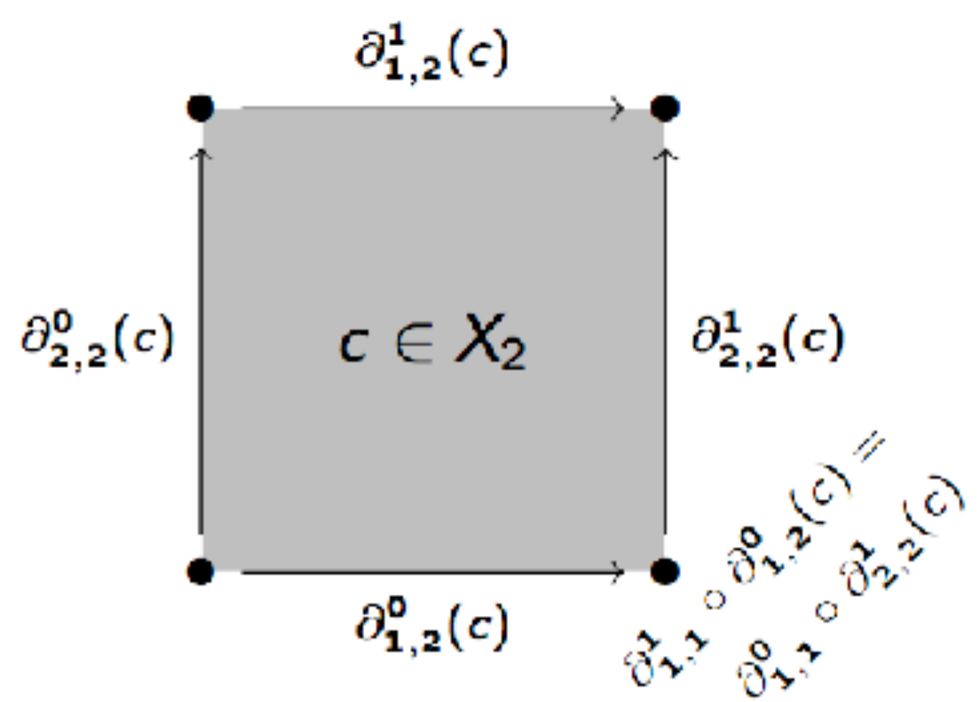
- a collection of sets $(X_n)_{n\in\mathbb{N}}$,
- a collection of functions $(\partial^{\alpha}_{i,n} : X_n \longrightarrow X_{n-1})_{n>0,\, 1\leq i\leq n,\, \alpha\in\{0,1\}}$

satisfying for $i > j$,

$$\partial^{\beta}_{j,n} \circ \partial^{\alpha}_{i,n+1} = \partial^{\alpha}_{i-1,n} \circ \partial^{\alpha}_{j,n+1}$$

Directed graph:

- $X_0 =$ **set of vertices,**
- $X_1 =$ **set of edges,**
- $X_{n>1} = \emptyset$**,**
- $\partial^0_{1,1} =$ **source function,**
- $\partial^1_{1,1} =$ **target function,**
- **equations are trivial.**



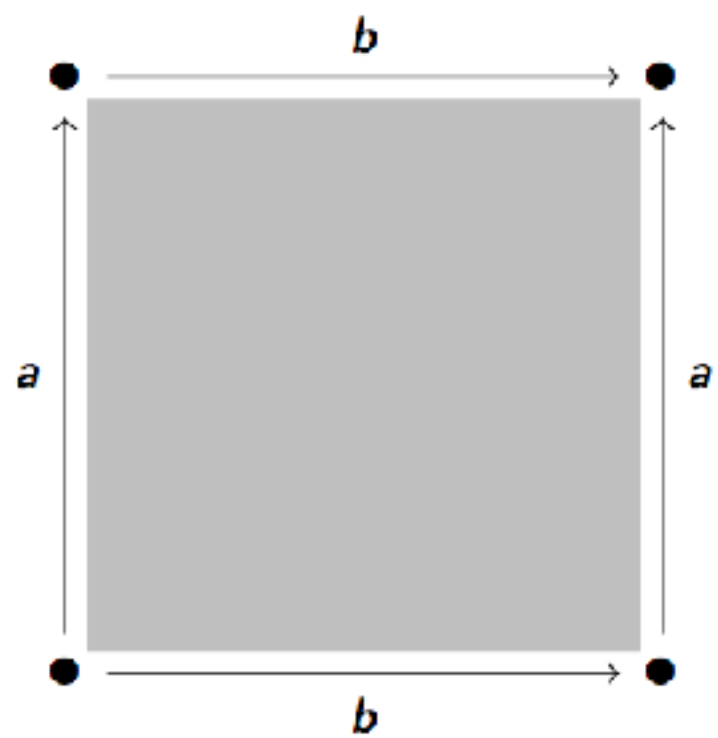
Higher Dimensional Automata [Pratt91]

A ***Higher Dimensional Automata*** **is:**

- **a precubical set** (X, ∂) **,**
- **an initial state** $i_0 \in X_0$**,**
- **a labelling function** $\lambda : X_1 \longrightarrow \Sigma$

satisfying for every $c \in X_2$**:**

$$\lambda(\partial^1_{i,2}(c)) = \lambda(\partial^0_{i,2}(c))$$



HDA, categorically

The category of HDA is equivalent to the following double slice category of the category of presheaves over the cube category:

$$1/[\,\square^{op}, Set]/\Sigma$$

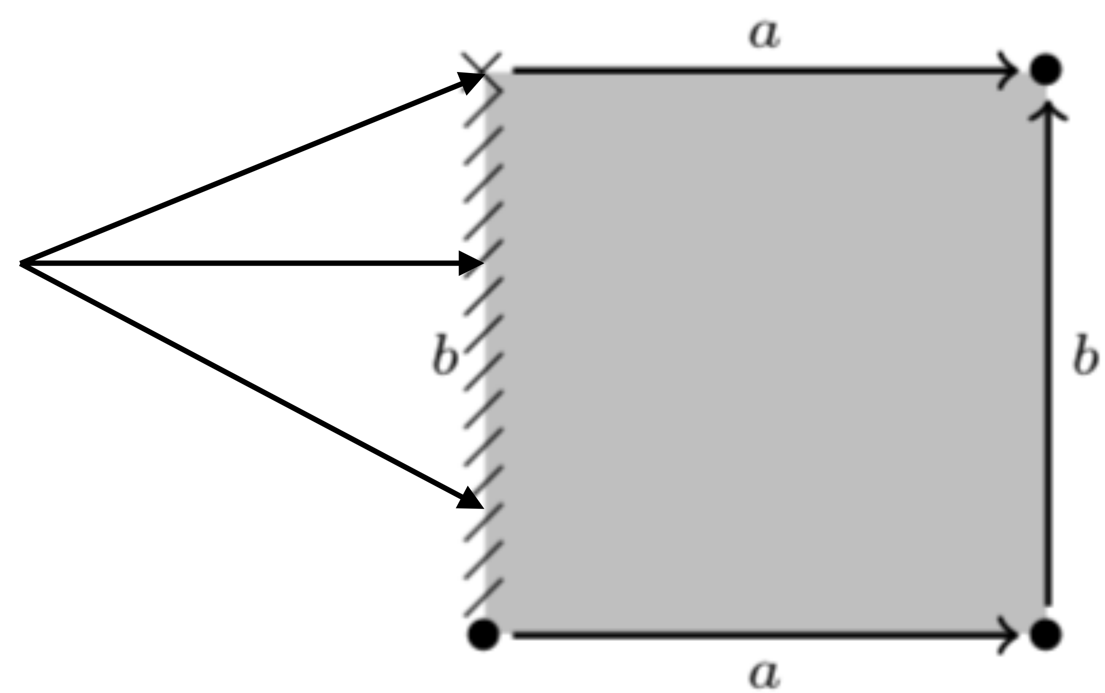
Allowing partiality

Category theory wins

Extending HDA?

How would you model:
"Actions *a* and *b* can be done at the same time,
but *a* must start before *b*."?

States where *b* has started
but not *a*.
Those must be removed.



In an elegant way

The category of partial HDA is the following double slice category of the category of lax functors over the cube category:

$$1/Lax(\square^{op}, pSet)/\Sigma$$

In short: replace some functions (corresponding to the face maps) by partial functions

In an ugly way

A ***partial precubical set*** is:

- a collection of sets $(X_n)_{n\in\mathbb{N}}$,
- a collection of partial functions $(\partial^{\alpha_1,\ldots,\alpha_k}_{i_1<\ldots<i_k,n} : X_n \longrightarrow X_{n-k})_{n>0,\, 1\leq k\leq n,\, \alpha_j\in\{0,1\}}$

satisfying for $i > j$,

$$\partial^{\beta_1,\ldots,\beta_n}_{j_1<\ldots<j_n} \circ \partial^{\alpha_1,\ldots,\alpha_m}_{i_1<\ldots<i_m} \subseteq \partial^{\gamma_1,\ldots,\gamma_p}_{k_1<\ldots<k_p}$$

A ***partial Higher Dimensional Automata*** is:

- a partial precubical set (X, ∂),
- an initial state $i_0 \in X_0$,
- a collection of labelling functions $(\lambda_n : X_n \longrightarrow \Sigma^n)_{n\in\mathbb{N}}$

satisfying for every $c \in X_n$:

$$\lambda_{n-1}(\partial^1_{i,n}(c)) = \lambda_{n-1}(\partial^0_{i,n}(c))$$

So… now what?

- **We have a *geometric* model of true concurrency.**

$$HDA, pHDA \longrightarrow Top, dTop$$

- **We can use tools from mathematics (algebraic topology) to study those models.**

- **This gave rise to a new mathematical field, the**

“directed algebraic topology”

Come to see my poster to see what this is, and what we can do with algebraic topology in true concurrency!