Fixed-Point Theorems for Non-Transitive Relations

54th TRS meeting
15th-17th of March 2021

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Introduction

- Interactive Theorem Proving is appreciated for reliability
- But it is also engineering tool for mathematics (esp. Isabelle/jEdit)
    - refactoring proofs and claims
    - sledgehammer
    - quickcheck/nitpick(/nunchaku)
    - locales for organising concepts
- We develop an Isabelle library of **order theory** (as a case study)
  $\Rightarrow$ we could generalise many known results, like:
    - completeness conditions: duality and relationships
    - fixed-point theorems for monotone functions (Knaster-Tarski, and many others)
    - fixed-point theorems for inflationary functions (Bourbaki-Witt)
    - iterative fixed-point theorems for monotone functions (Kleene, and others)

Why giving this talk at a TRS meeting?

- Order theory is just a case study, the methodology could also be used for rewriting relations
- Some concepts of order theory make sense for rewriting relations
  - confluence ≡ existence of bounds
  - Knuth-Bendix completion ≡ adding sups in the relation
  - termination ≡ the reverse relation is well-founded
- Complete relations and sups are crucial for infinite term rewriting
- Getting rid of assumptions can be useful for more involved techniques

References

- A. Yamada, J. Dubut. Complete Non-Orders and Fixed Points. *In* ITP'19.
  https://drops.dagstuhl.de/opus/volltexte/2019/11085/pdf/LIPIcs-ITP-2019-30.pdf
- J. Dubut, A. Yamada. Fixed Point Theorems for Non-Transitive Relations. LMCS. To appear.
  https://arxiv.org/pdf/2009.13065.pdf
- Entry in the Archive of Formal Proofs
  https://www.isa-afp.org/entries/Complete_Non_Orders.html

Partial order

- **set** (A) **+ binary relation** $(\sqsubseteq)$
- **reflexive** $\Leftrightarrow x \sqsubseteq x$
- **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$
- **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

Partial order

- **set** (A) **+ binary relation** $(\sqsubseteq)$

```
locale related_set =
  fixes A :: "'a set" and less_eq :: "'a ⇒ 'a ⇒ bool" (infix "⊑" 50)
```

- **reflexive** $\Leftrightarrow x \sqsubseteq x$

```
locale reflexive = related_set +
  assumes refl[intro]: "x ∈ A ⟹ x ⊑ x"
```

- **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$

```
locale transitive = related_set +
  assumes trans[trans]: "x ⊑ y ⟹ y ⊑ z ⟹ x ∈ A ⟹ y ∈ A ⟹ z ∈ A ⟹ x ⊑ z"
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- **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

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locale antisymmetric = related_set +
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Partial order

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$\boxed{\textbf{locale}\ \text{partially\_ordered\_set} = \text{reflexive} + \text{transitive} + \text{antisymmetric}}$

Quasi order

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- **transitive** $\Leftrightarrow x \sqsubseteq$ … $\sqsubseteq z$

  locale transiti

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$\boxed{\textbf{locale}\ \text{quasi\_ordered\_set} = \text{reflexive} + \text{transitive}}$

sublocale partially_ordered_set ⊆ quasi_ordered_set

Pseudo order (Skala 71)

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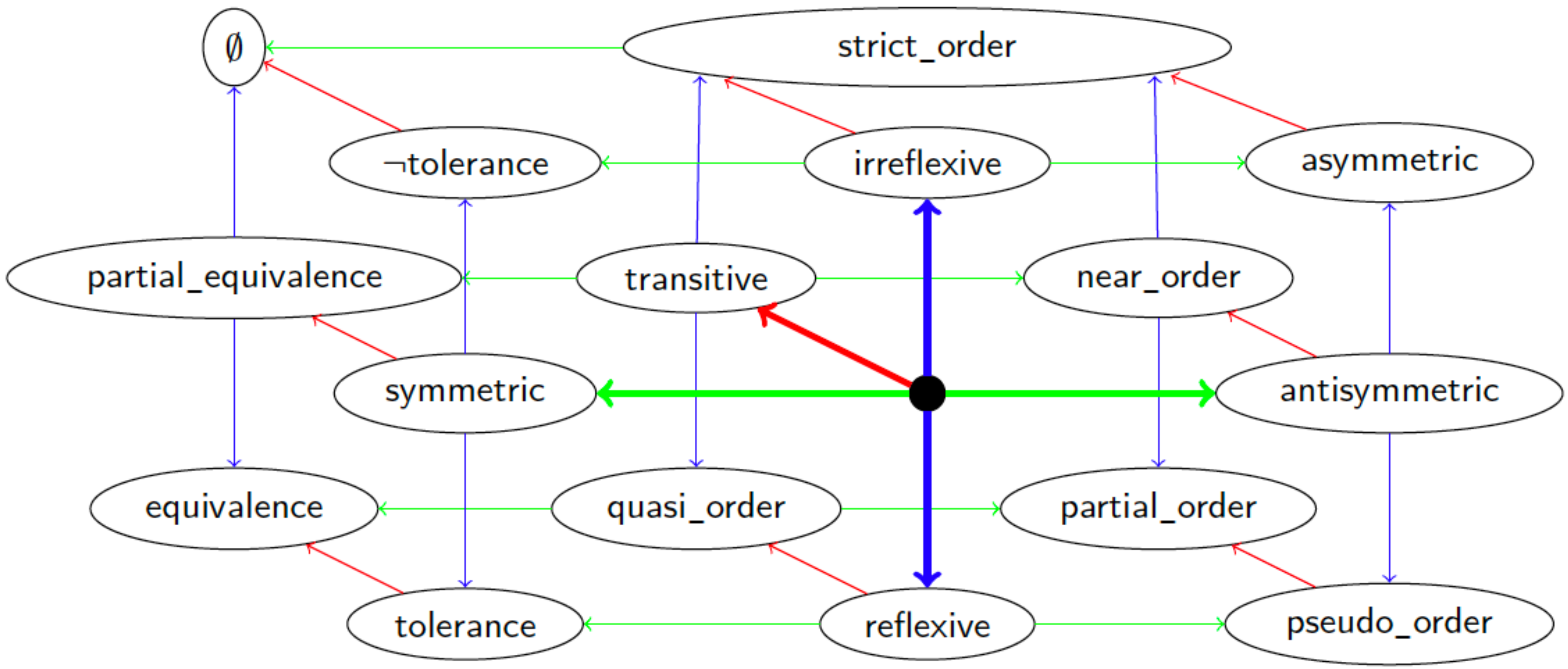
locale pseudo_ordered_set = reflexive + antisymmetric
sublocale partially_ordered_set ⊆ pseudo_ordered_set

Hierarchy of locales



You can easily add termination, confluence, … into this picture

FP theorems

- Data:
  - A relation $(A, \sqsubseteq)$
  - A function $f : A \to A$
- Assumptions:
  - $(A, \sqsubseteq)$ is an "order"
  - f interacts with $\sqsubseteq$
  - Some sups for $\sqsubseteq$ exist (completeness)
- Conclusions:
  - There exists a fixpoint $f(x) = x$
  - There is a least fixpoint
  - The set of fixpoints is complete

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Usually: partial order
Here: pseudo order or even less

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Monotone, inflationary, continuous

- Monotone
  $$\forall x, y \in A, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$

- Inflationary
  $$\forall x \in A, x \sqsubseteq f(x)$$

- Continuous
  $$f(\sup\ C) = \sup\,(f(C)) \text{ for some } C$$
  Example 1: if true for all C omega chains, f is omega continuous
  Example 2: if true for all directed sets, f is Scott continuous

Monotone, inflationary, continuous

Ex: Knaster-Tarski

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Ex: Kleene

FP theorems

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  - f interacts with $\sqsubseteq$
  - Some sups for $\sqsubseteq$ exist (completeness)
- Conclusions:
  - There exists a fixpoint $f(x) = x$
  - There is a least fixpoint
  - The set of fixpoints is complete

Extreme bounds for relations

- **b is a bound of X for $\sqsubseteq$** $\Leftrightarrow \forall x \in X, x \sqsubseteq b$

  definition "bound X $(\sqsubseteq)$ $b \equiv \forall x \in X.\ x \sqsubseteq b$"

- **b is <u>an</u> extreme element of X for $\sqsubseteq$** $\Leftrightarrow$
  $b \in X$ and $\forall x \in X, x \sqsubseteq b$

  definition "extreme X $(\sqsubseteq)$ $e \equiv e \in X \wedge (\forall x \in X.\ x \sqsubseteq e)$"

- **b is <u>an</u> extreme bound of X for $\sqsubseteq$** $\Leftrightarrow$
  b is a bound of X for $\sqsubseteq$ and
  for all bounds b' of X for $\sqsubseteq$, $b \sqsubseteq b'$

  abbreviation "extreme_bound A $(\sqsubseteq)$ $X \equiv$ extreme $\{b \in A.\ \text{bound } X\ (\sqsubseteq)\ b\}$ $(\sqsupseteq)$"

Extreme bounds for relations

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No need for order assumptions...

… but beware of counter-intuitions

- An extreme bound of the singleton $\{x\}$:
  - may not exist
  - or if it exists, it may not be equal to x itself

- Extreme elements and extreme bound are not unique

  If c and d are extreme elements of the same set then $c \sim d$, that is $c \sqsubseteq d$ and $d \sqsubseteq c$

- c being an (extreme) bound of X and $c \sqsubseteq d$ do not imply that d is a bound of X

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Equivalent to reflexivity

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Equivalent to reflexivity

- Extreme e

  If c an … $c \sqsubseteq d$ and $d \sqsubseteq c$

Equivalent to antisymmetry

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Equivalent to antisymmetry

- c being an (… imply that d is a b

Equivalent to transitivity

What about rewriting relations?

- **Confluence using bounds:**
  $\sqsubseteq$ is confluent iff every set $\{x \sqsubseteq y, z\}$ has a bound

- **Extreme elements are normal forms**

- **What about extreme bounds?**

Completeness assumptions

- **Knaster-Tarski:** all the subsets have a sup
- **Bourbaki-Witt:** all the chains have a sup
- **Pataraia:** all the directed sets have a sup
- **Kleene:** all the omega chain have a sup

Generalisation and instantiation

- **A relation $(A, \sqsubseteq)$ is C-complete, for C a class of sets, if all subsets of A in C has an extreme bound**

```
definition complete ("_-complete" [999]1000) where
  "𝒞-complete A (⊑) ≡ ∀X ⊆ A. X ∈ 𝒞 ⟶ (∃s. extreme_bound A (⊑) X s)"
```

- **Chains:**

```
locale connex = related_set +
  assumes "x ∈ A ⟹ y ∈ A ⟹ x ⊑ y ∨ y ⊑ x"
```

- **Directed sets:**

```
definition "directed X (⊑) ≡ ∀x ∈ X. ∀y ∈ X. ∃z ∈ X. x ⊑ z ∧ y ⊑ z"
```

- **Omega chains:**

```
definition "omega_chain X (⊑) ≡ X ∈ {range c | c :: nat ⇒ 'a. monotone (≤) (⊑) c}"
```

Completeness assumptions

- **Knaster-Tarski:** all the subsets have a sup

“UNIV-complete A $(\sqsubseteq)$”

- **Bourbaki-Witt:** all the chains have a sup

“{X. connex X $(\sqsubseteq)$}-complete A $(\sqsubseteq)$”

- **Pataraia:** all the directed sets have a sup

“$\{X$. directed X $(\sqsubseteq)\}$-complete A $(\sqsubseteq)$”

- **Kleene:** all the omega chain have a sup

“{X. omega_chain X $(\sqsubseteq)$}-complete A $(\sqsubseteq)$”

Completeness in TRS: Knuth-Bendix

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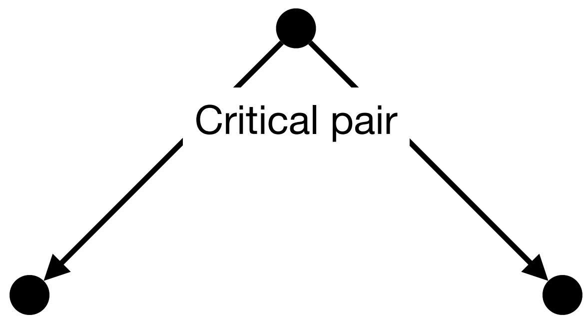
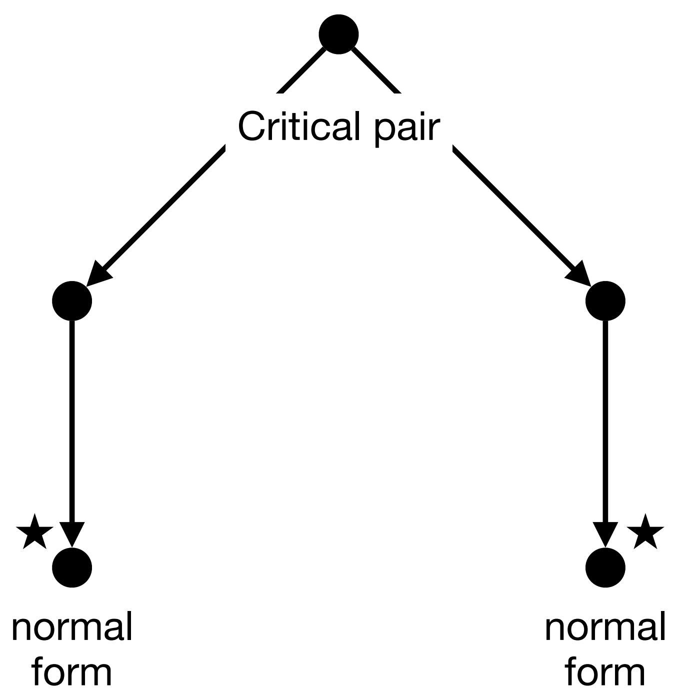
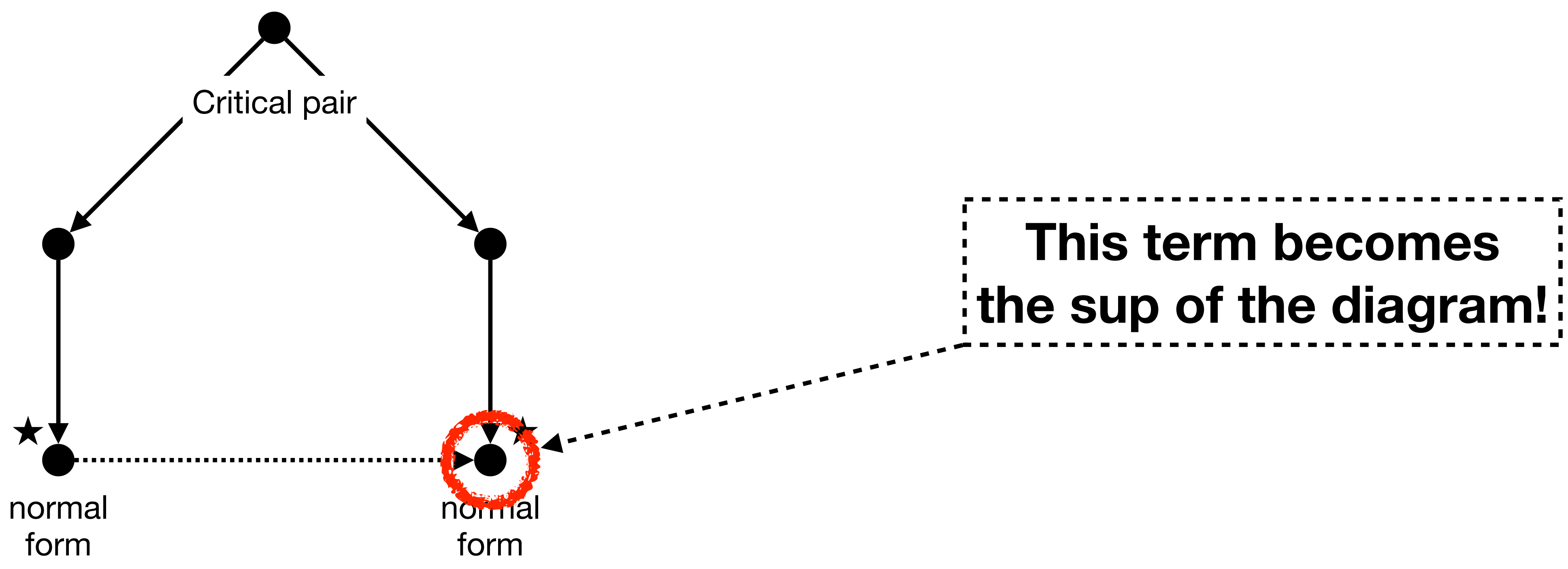
Completeness in TRS: Knuth-Bendix



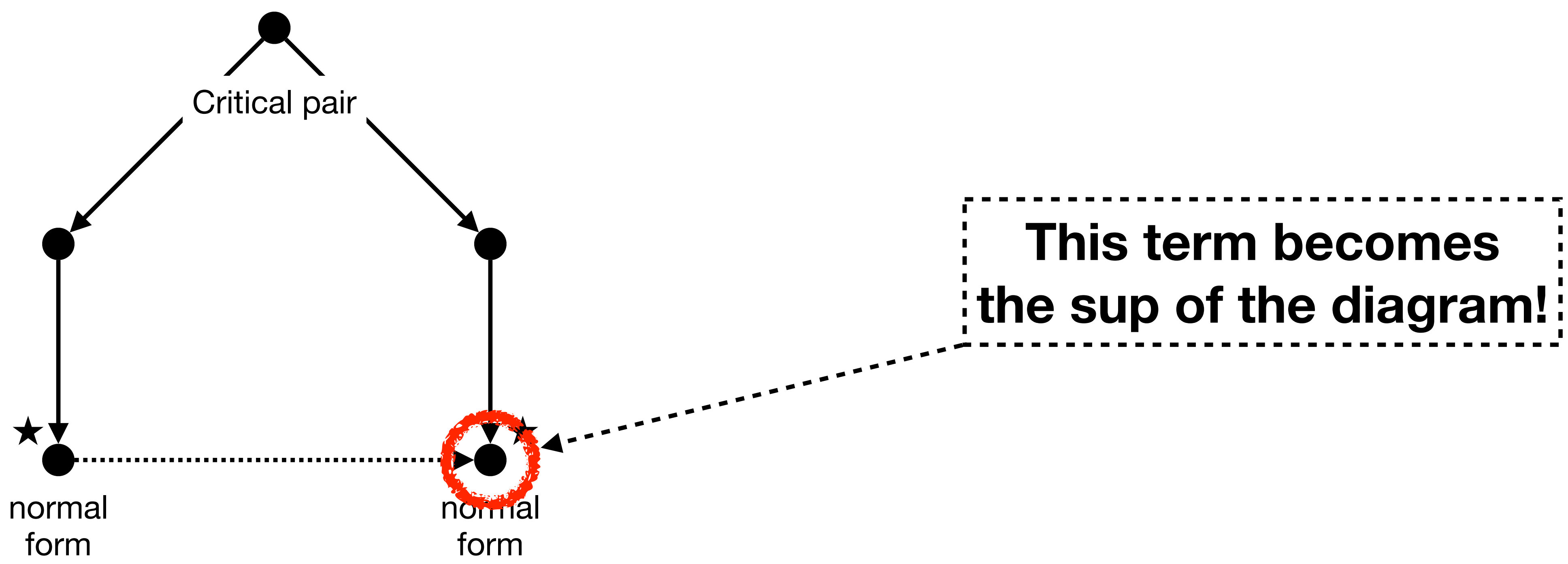
Completeness in TRS: Knuth-Bendix



Completeness in TRS: Knuth-Bendix



Completeness in TRS: Knuth-Bendix



Knuth-Bendix completion
=
Making the rewriting relation complete w.r.t. those diagrams

Well-orderedness

- **Well-related set:** every non-empty subset has a least element

$$\textsf{locale well\_related\_set} = \textsf{related\_set} + \\ \textsf{assumes } \text{“}X \subseteq A \Longrightarrow X \neq \{\} \Longrightarrow \exists e.\ \textsf{extreme } X\ (\sqsupseteq)\ e\text{”}$$

- **Well-related sets are chains:**

$$\textsf{sublocale well\_related\_set} \subseteq \textsf{connex}$$

- **Well-related sets are well-founded:**

$$\text{“}\forall a \in A.\ (\forall x \in A.\ (\forall y \in A.\ y \sqsubset x \longrightarrow P\ y) \longrightarrow P\ x) \longrightarrow P\ a\text{”}$$

$$x \sqsubset y \ \equiv\ x \sqsubseteq y \wedge y \not\sqsubseteq x$$

- **Completeness w.r.t. well-related sets is enough!**

Well-orderedness

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No need for order assumptions

FP theorems for monotone maps

paper	reflexivity	transitivity	antisym.	complete w.r.t.	existence	least	complete	ordinals	axiom of choice
Knaster Tarski	✓	✓	✓	UNIV	✓	✓	✓		
Abian Brown	✓	✓	✓	wf chains	✓	X	X		
Markowsy	✓	✓	✓	chains	✓	✓	✓	✓	
Pataraia	✓	✓	✓	directed sets	✓	✓	✓		
Bhatta George	✓	X	✓	wf chains	✓	✓	✓	✓	
Stouti Maaden	✓	X	✓	UNIV	✓	✓	X		
Grall	✓	✓	✓	chains	✓	✓	✓		✓
DY v.1	X	X	✓	UNIV	✓	✓	✓		
DY v.2	X	X	✓	well-related⊆C & closure prop.	✓	✓	✓		

FP theorems for inflationary maps

paper	reflexivity	transitivity	antisym.	complete w.r.t.	existence	ordinals	axiom of choice
Bourbaki Witt	✓	✓	✓	chains	✓	✓	
Abian Brown	✓	✓	✓	wf chains	✓		
Grall	✓	✓	✓	chains	✓		✓
DY v.1	X	X	✓	UNIV	✓		
DY v.2	✓	X	✓	well-related sets	✓		

Iterative FP theorems

paper	reflexivity	transitivity	antisym.	continuity	complete w.r.t.	existence	least
Kantorovitch	✓	✓	✓	omega	UNIV	✓	✓
Tarski	✓	✓	✓	countably distributive	countable sets	✓	✓
Kleene	✓	✓	✓	Scott	directed sets	✓	✓
Mashburn	✓	✓	✓	omega	omega chains	✓	✓
DY	X	X	✓	omega	omega chains	✓	✓

Why antisymmetry cannot be avoided?

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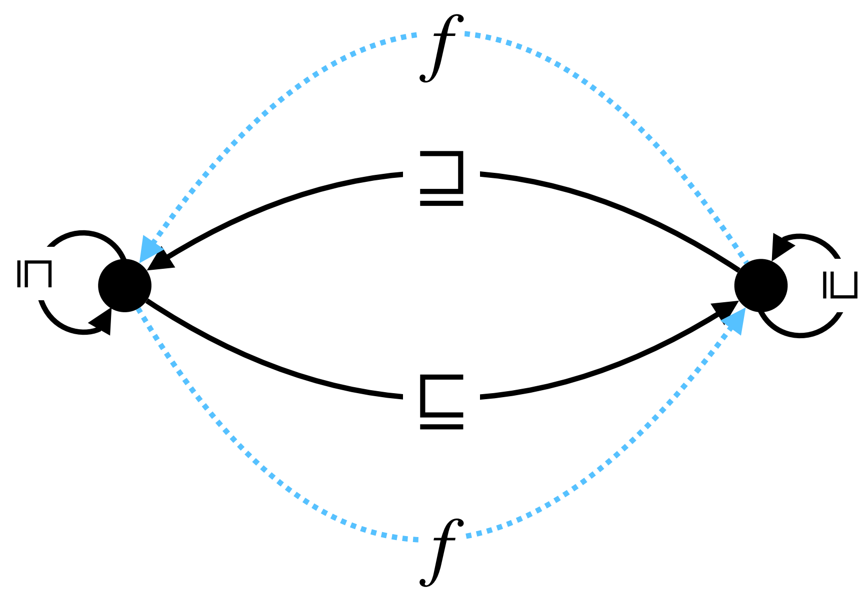


$\sqsubseteq$ is complete reflexive and transitive,
f is monotone, inflationary and continuous,
but no fixpoints

Why antisymmetry cannot be avoided?

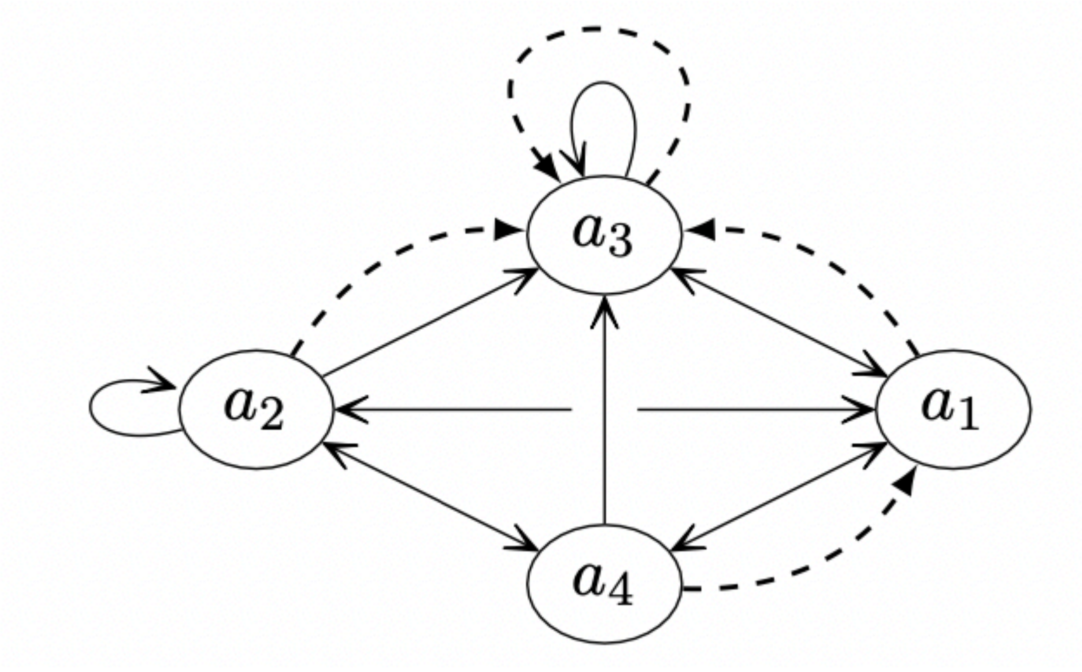
Every proof in the literature prove

the existence of a (least) quasi fixpoint

$x \sim f(x)$**, i.e.,** $x \sqsubseteq f(x)$ **and** $f(x) \sqsubseteq x$

and use antisymmetry to conclude it is a fixpoint

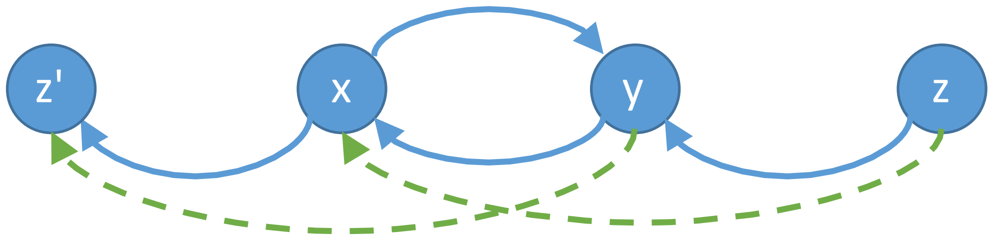
Can we avoid antisymmetry?



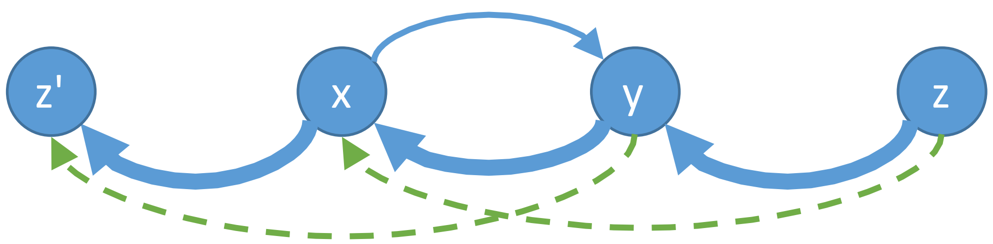
$\sqsubseteq$ is complete, f is monotone,
but no least fixpoints

Attractivity

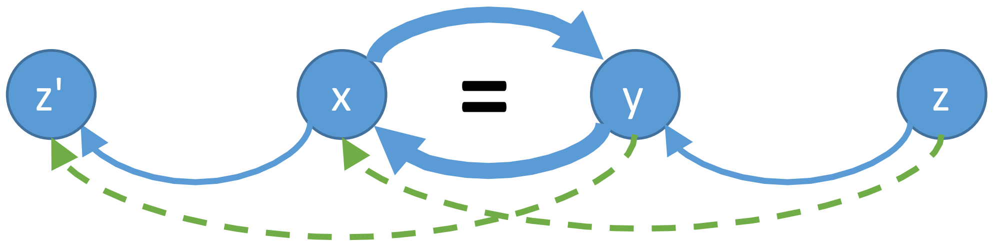
If $x \sim y$ and $z \sqsubseteq y$ then $z \sqsubseteq x$
If $x \sim y$ and $x \sqsubseteq z'$ then $y \sqsubseteq z'$



sublocale transitive $\subseteq$ attractive



sublocale antisymmetric $\subseteq$ attractive



Without antisymmetry

attractivity	map	Complete w.r.t.	existence	least	complete
X	monotone	UNIV	✓		
X	inflationary	UNIV	✓		
✓	monotone	UNIV	✓	✓	✓
✓	monotone	well-related sets	✓	✓	
✓	monotone	well-related $\subseteq$ C & closure prop.	✓	✓	✓
✓	omega continuous	omega chains	✓	✓	

Without antisymmetry

attractivity	map	Complete w.r.t.	existence	least	complete
X	monotone	UNIV	✓		
X	inflationary	UNIV	✓		
✓	monotone	UNIV	✓	✓	✓
✓	monotone	well-related sets	✓	✓	
✓	monotone	well-related $\subseteq$ C & closure prop.	✓	✓	✓
✓	omega continuous	omega chains	✓	✓	

Main statement

theorem attract_mono_imp_fp_qfp_complete:
  assumes "attractive A ($\sqsubseteq$)" **and** "$\mathcal{C}$-complete A ($\sqsubseteq$)"
    and "$\forall X \subseteq A$. well_related_set X ($\sqsubseteq$) $\longrightarrow X \in \mathcal{C}$"
    and extend: "$\forall X \in \mathcal{C}.\ \forall Y \in \mathcal{C}.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in \mathcal{C}$"
    and "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f" **and** "$P \subseteq \{x \in A.\ f\ x = x\}$"
  shows "$\mathcal{C}$-complete ($\{q \in A.\ f\ q \sim q\} \cup P$) ($\sqsubseteq$)"

Main statement

Only order-theoretic assumption

theorem attract_mono_imp_fp_qfp_complete:
  assumes $\boxed{\text{“attractive } A\ (\sqsubseteq)\text{”}}$ **and** “$\mathcal{C}$-complete $A\ (\sqsubseteq)$”
    and “$\forall X \subseteq A.$ well_related_set $X\ (\sqsubseteq) \longrightarrow X \in \mathcal{C}$”
    and extend: “$\forall X \in \mathcal{C}.\ \forall Y \in \mathcal{C}.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in \mathcal{C}$”
    and “monotone_on $A\ (\sqsubseteq)\ (\sqsubseteq)\ f$” **and** “$P \subseteq \{x \in A.\ f\ x = x\}$”
  shows “$\mathcal{C}$-complete $(\{q \in A.\ f\ q \sim q\} \cup P)\ (\sqsubseteq)$”

Main statement

theorem attract_mono_imp_fp_qfp_complete:
  assumes "attractive A $(\sqsubseteq)$" **and** "$\mathcal{C}$-complete A $(\sqsubseteq)$"
    and "$\forall X \subseteq A$. well_related_set X $(\sqsubseteq) \longrightarrow X \in \mathcal{C}$"
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    and "monotone_on A $(\sqsubseteq)$ $(\sqsubseteq)$ f" **and** "$P \subseteq \{x \in A.\ f\ x = x\}$"
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Monotone case

Main statement

theorem attract_mono_imp_fp_qfp_complete:
  assumes "attractive A ($\sqsubseteq$)" **and** "$\mathcal{C}$-complete A ($\sqsubseteq$)"
    and $\boxed{\text{"}\forall X \subseteq A.\ \text{well\_related\_set}\ X\ (\sqsubseteq) \longrightarrow X \in \mathcal{C}\text{"}}$
    and $\boxed{\text{extend: "}\forall X \in \mathcal{C}.\ \forall Y \in \mathcal{C}.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in \mathcal{C}\text{"}}$
    and "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f" **and** "$P \subseteq \{x \in A.\ f\ x = x\}$"
  shows "$\mathcal{C}$-complete ($\{q \in A.\ f\ q \sim q\} \cup P$) ($\sqsubseteq$)"

True for all classes C mentioned here

Step 1: defining derivations

Assumptions: None

Idea: derivation trees [Grall'10]

$$\frac{x \text{ is derivable}}{f(x) \text{ is derivable}}\ (\textbf{Succ}) \qquad \frac{C \text{ chain and } \forall x \in C, x \text{ is derivable}}{\sup(C) \text{ is derivable}}\ (\textbf{Lim})$$

Intuition: the derivable elements are of the form $f^\alpha(\perp)$ for some ordinal α

Proof:

1. $D = \{x \,.\, x \text{ is derivable}\}$ is a chain, and $c = \sup(D)$
2. $c \sqsubseteq f(c)$ (easy proof)
3. c is derivable (by **Lim**)
4. $f(c)$ is derivable (by **Succ**), and $f(c) \sqsubseteq c$

Idea: derivation trees [Grall'10]

$$\frac{x \text{ is derivable}}{f(x) \text{ is derivable}}\ (\textbf{Succ}) \qquad \frac{C \text{ chain and } \forall x \in C, x \text{ is derivable}}{\sup(C) \text{ is derivable}}\ (\textbf{Lim})$$

Intuition: the derivable elements are of the form $f^{\alpha}(\perp)$ for some ordinal α

Proof:

1. $D = \{x \,.\, x \text{ is derivable}\}$ is a chain $\boxed{\text{This easily fails when } \sqsubseteq \text{ is not an order}}$
2. $c \sqsubseteq f(c)$ (easy proof)
3. c is derivable (by **Lim**)
4. $f(c)$ is derivable (by **Succ**), and $f(c) \sqsubseteq c$

Idea: derivation trees [Grall'10]

$$\frac{x \text{ is derivable}}{f(x) \text{ is derivable}}\ (\textbf{Succ}) \qquad \frac{C \text{ chain and } \forall x \in C, x \text{ is derivable}}{\sup(C) \text{ is derivable}}\ (\textbf{Lim})$$

Intuition: the derivable elements are of the form $f^{\alpha}(\bot)$ for some ordinal α

Proof:

1. $D = \{x \,.\, x \text{ is derivable}\}$ is a chain **This easily fails when $\sqsubseteq$ is not an order**
2. $c \sqsubseteq f(c)$ (easy proof)
3. c is derivable (by **Lim**) **This uses the axiom of choice**
4. $f(c)$ is derivable (by **Succ**), and $f(c) \sqsubseteq c$

Care about derivations, not derivables

definition "derivation $X \equiv X \subseteq A \wedge$ well_ordered_set X $(\sqsubseteq) \wedge$
$(\forall x \in X.$ let $Y = \{y \in X.\ y \sqsubset x\}$ in
$(\exists y.$ extreme Y $(\sqsubseteq)$ $y \wedge x = f\ y) \vee (f\ `Y \subseteq Y \wedge$ extreme_bound A $(\sqsubseteq)$ Y $x))$"

Care about derivations, not derivables

A derivation tree is well-founded

definition "derivation $X \equiv X \subseteq A \wedge \boxed{\text{well\_ordered\_set } X\ (\sqsubseteq)} \wedge$
$(\forall x \in X.$ let $Y = \{y \in X.\ y \sqsubset x\}$ in
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Care about derivations, not derivables

A derivation tree is well-founded

Ancestors of x in the derivation

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$\boxed{(\exists y.\ \text{extreme } Y\ (\sqsubseteq)\ y \wedge x = f\ y)} \vee (f\ `Y \subseteq Y \wedge \text{extreme\_bound } A\ (\sqsubseteq)\ Y\ x))$"

x is derived by the successor case

Care about derivations, not derivables

A derivation tree is well-founded

Ancestors of x in the derivation

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$\boxed{(\exists y.\ \text{extreme } Y\ (\sqsubseteq)\ y \wedge x = f\ y)} \vee \boxed{(f\ `\ Y \subseteq Y \wedge \text{extreme\_bound } A\ (\sqsubseteq)\ Y\ x)})$"

x is derived by the successor case

x is derived by the limit case

Care about derivations, not derivables

A derivation tree is well-founded

Ancestors of x in the derivation

```
definition "derivation X ⟺ X ⊆ A ∧ well_ordered_set X (⊑) ∧
  (∀x ∈ X. let Y = {y ∈ X. y ⊏ x} in
   (∃y. extreme Y (⊑) y ∧ x = f y) ∨ (f ` Y ⊆ Y ∧ extreme_bound A (⊑) Y x))"
```

x is derived by the successor case

x is derived by the limit case

Not in [Grall'10]

Step 2: contructing a quasi fixpoint using derivations

Assumptions:

- $\sqsubseteq$ is antisymmetric
- If x is derivable, then $f(x) \sqsubseteq f(x)$
- If x and y are derivable, and if $x \sqsubseteq y$ then $x \sqsubseteq f(y)$
- $\sqsubseteq$ is well-complete

Step 2: contructing a quasi fixpoint using derivations

Assumptions:

- $\sqsubseteq$ is antisymmetric
- If x is derivable, then $f(x) \sqsubseteq f(x)$ **True if $\sqsubseteq$ is reflexive**
- If x and y are derivable, and if $x \sqsubseteq y$ then $x \sqsubseteq f(y)$
- $\sqsubseteq$ is well-complete

Step 2: contructing a quasi fixpoint using derivations

Assumptions:

- $\sqsubseteq$ is antisymmetric
- If x is derivable, then $f(x) \sqsubseteq f(x)$ — **True if $\sqsubseteq$ is reflexive**
- If x and y are derivable, and if $x \sqsubseteq y$ then $x \sqsubseteq f(y)$ — **True if $\sqsubseteq$ is transitive and f is inflationary**
- $\sqsubseteq$ is well-complete

Goal

The main task is to prove:

definition "derivable $x \equiv \exists X.$ derivation $X \wedge x \in X$"

lemma derivation_derivable: "derivation $\{x.$ derivable $x\}$"

Indeed, together with (elementary proofs, 68 and 28 lines):

lemma derivable_closed:
  assumes "derivable x" **shows** "derivable $(f\ x)$"

lemma derivation_lim:
  assumes "derivation P" **and** "f ` P ⊆ P" **and** "extreme_bound A (⊑) P p"
  shows "derivation (P ∪{p})"

We can conclude the existence of a fixpoint (elementary proof, 14 lines):

lemma sup_derivable_fp:
  assumes "extreme_bound A (⊑) $\{x.$ derivable $x\}$ p"
  shows "$f\ p = p$"

Goal

The main task is to prove:

definition "derivable $x \equiv \exists X.$ derivation $X \wedge x \in X$"

$\boxed{\textbf{lemma}\ \text{derivation\_derivable: ``derivation } \{x.\ \text{derivable } x\}\text{''}}$

Indeed, together with (elementary proofs, 68 and 28 lines):

lemma derivable_closed:
  assumes "derivable x" **shows** "derivable $(f\ x)$"

lemma derivation_lim:
  assumes "derivation P" **and** "f ` P $\subseteq$ P" **and** "extreme_bound A ($\sqsubseteq$) P p"
  shows "derivation (P $\cup$\{p\})"

We can conclude the existence of a fixpoint (elementary proof, 14 lines):

lemma sup_derivable_fp:
  assumes "extreme_bound A ($\sqsubseteq$) $\{x.$ derivable $x\}$ p"
  shows "$f\ p = p$"

Derivables are well-related

```
interpretation derivable: well_ordered_set "{x. derivable x}" "(⊑)"
```

The main trick to avoid the axiom of choice is to use:

```
lemma closed_UN_well_founded:
  assumes "∀X ∈ 𝒳. well_founded X (⊏) ∧ (∀x ∈ X. ∀y ∈ ⋃𝒳. y ⊏ x ⟶ y ∈ X)"
  shows "well_founded (⋃𝒳) (⊏)"
```

With the collection $\mathscr{X}$ of derivations.

Derivables are well-related

interpretation derivable: well_ordered_set "$\{x.$ derivable $x\}$" "$(\sqsubseteq)$"

The main trick to avoid the axiom of choice is to use:

lemma closed_UN_well_founded:
assumes "$\forall X \in \mathcal{X}.$ well_founded X $(\sqsubset) \wedge \boxed{\forall x \in X.\ \forall y \in \bigcup \mathcal{X}.\ y \sqsubset x \longrightarrow y \in X}$"
shows "well_founded $(\bigcup \mathcal{X})$ $(\sqsubset)$"

With the collection $\mathcal{X}$ of derivations.

lemma derivations_cross_compare:
assumes "derivation X" **and** "derivation Y" **and** "$x \in X$" **and** "$y \in Y$"
shows "$(x \sqsubset y \wedge x \in Y) \vee x = y \vee (y \sqsubset x \wedge y \in X)$"

Derivables are well-related

interpretation derivable: well_ordered_set "$\{x.$ derivable $x\}$" "$(\sqsubseteq)$"

The main trick to avoid the axiom of choice is to use:

lemma closed_UN_well_founded:
  assumes "$\forall X \in \mathcal{X}.$ well_founded X $(\sqsubset) \wedge$ $\boxed{\forall x \in X.\ \forall y \in \bigcup \mathcal{X}.\ y \sqsubset x \longrightarrow y \in X)}$"
  shows "well_founded $(\bigcup \mathcal{X})$ $(\sqsubset)$"

With the collection $\mathscr{X}$ of derivations.

lemma derivations_cross
  assumes "derivatio… …ation Y" **and** "$x \in X$" **and** "$y \in Y$"
  shows "$(x \sqsubset y$ … $y \vee (y \sqsubset x \wedge y \in X)$"

Main lemma
Not in [Grall'10]

Main lemma

lemma derivations_cross_compare:
  assumes "derivation X" **and** "derivation Y" **and** "$x \in X$" **and** "$y \in Y$"
  shows "$(x \sqsubset y \wedge x \in Y) \vee x = y \vee (y \sqsubset x \wedge y \in X)$"

Hard proof (204 lines), by double induction on x and y and case distinctions

Some arguments rely on "useful" derivations used by [Grall'10] (83 lines from the 210):

lemma derivation_useful:
  assumes "derivation X" **and** "$x \in X$" **and** "$y \in X$" **and** "$x \sqsubset y$"
  shows "$f\ x \sqsubseteq y$"

Step 3: proving leastness using monotonicity

Assumptions:

- $\sqsubseteq$ is antisymmetric
- f is monotone
- $\sqsubseteq$ is well-complete

Monotonicity implies the assumptions

Trivial (9 lines): a derivation is well_related so reflexive

lemma mono_imp_derivation_f_refl:
assumes "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f"
shows "$\forall X\ x.$ derivation $X \longrightarrow x \in X \longrightarrow f\ x \sqsubseteq f\ x$"

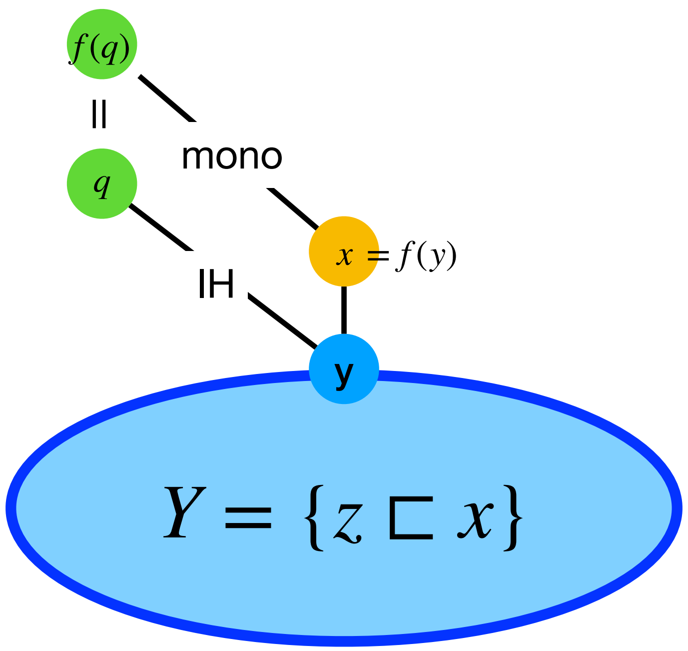
Easy (35 lines): by induction on x and case distinction

lemma mono_imp_derivation_infl:
assumes "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f"
shows "$\forall X\ x\ y.$ derivation $X \longrightarrow x \in X \longrightarrow y \in X \longrightarrow x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$"

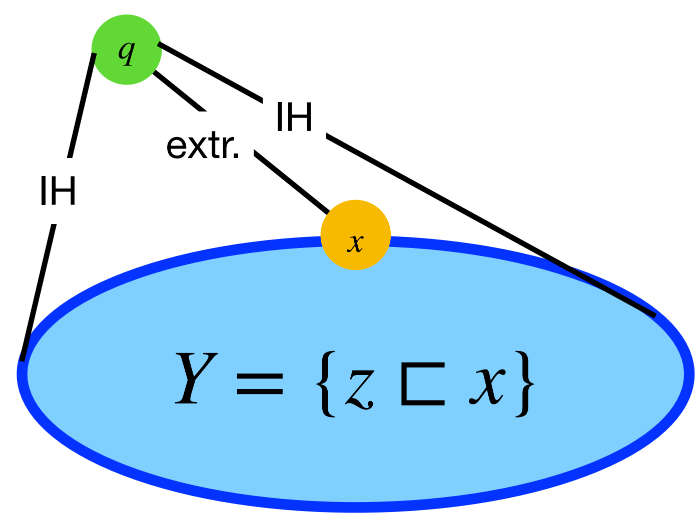
Existence of a least fixpoint

lemma mono_imp_ex_least_fp:
  assumes "well_complete A ($\sqsubseteq$)" **and** "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f"
  shows "$\exists p$. extreme $\{q \in A.\ f\ q = q\}$ ($\sqsupseteq$) p"

Proof (31 lines): By induction on derivable x, x is below any fixpoint q.



(Succ case)



(Lim case)

Step 4: from antisymmetry to attractivity

Assumptions:

- $\sqsubseteq$ is attractive
- f is monotone
- $\sqsubseteq$ is well-complete

Existence of the least quasi fixpoint, with attractivity

lemma attract_mono_imp_least_qfp:
  assumes "attractive A ($\sqsubseteq$)" **and** "well_complete A ($\sqsubseteq$)" **and** "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f"
  shows "$\exists c.$ extreme $\{p \in A.\ f\ p \sim p \vee f\ p = p\}$ ($\sqsupseteq$) $c \wedge f\ c \sim c$"

Proof (83 lines): Apply the previous step on a quotient of A

- **define** ecl ("[_]$_\sim$") **where** "$[x]_\sim \equiv \{y \in A.\ x \sim y\} \cup \{x\}$" **for** x
  define Q **where** "$Q \equiv \{[x]_\sim \mid.\ x \in A\}$"
  definition "$X \sqsubseteq^s Y \equiv \forall x \in X.\ \forall y \in Y.\ x \sqsubseteq y$"
  $(Q, \sqsubseteq^s)$ is antisymmetric and well-complete
- **define** F **where** "$F\ X \equiv \{y \in A.\ \exists x \in X.\ y \sim f\ x\} \cup f\ `X$" **for** X
  $F :\ Q \longrightarrow Q$ is well defined and monotone
- F then has a least fixpoint
- Every fixpoint of F is the class of a (quasi)-fixpoint

Step 5: completeness

Assumptions:

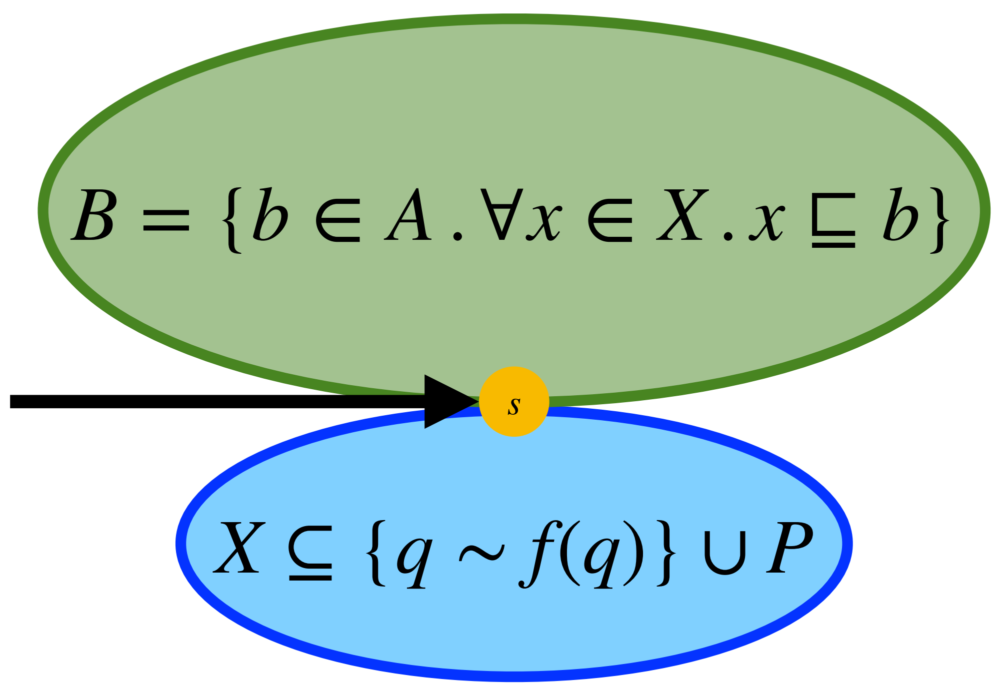
- $\sqsubseteq$ is attractive
- f is monotone
- $\sqsubseteq$ is C-complete with:
  - C containing well-related sets
  - C closed under ordered unions

General completeness

theorem attract_mono_imp_fp_qfp_complete:
  assumes "attractive A ($\sqsubseteq$)" **and** "$\mathcal{C}$-complete A ($\sqsubseteq$)"
    and "$\forall X \subseteq A$. well_related_set X ($\sqsubseteq$) $\longrightarrow X \in \mathcal{C}$"
    and extend: "$\forall X \in \mathcal{C}.\ \forall Y \in \mathcal{C}.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in \mathcal{C}$"
    and "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f" **and** "$P \subseteq \{x \in A.\ f\ x = x\}$"
  shows "$\mathcal{C}$-complete ($\{q \in A.\ f\ q \sim q\} \cup P$) ($\sqsubseteq$)"

Proof (75 lines):

The least fixpoint of f on B obtained by applying the previous step on B



B satisfies the assumptions of the previous step:

- B is f-closed (easy)
- B is well-complete:
  - Take $Y \subseteq B$ with $Y \in C$
  - $X \cup Y \in C$ by extend
  - $\sup_A(X \cup Y)$ exists
  - $\sup_A(X \cup Y) = \sup_B(Y)$

Instances: [Knaster-Tarski'55] & [Stouti-Maaden'13]

theorem (**in** antisymmetric) mono_imp_fp_complete:
  assumes "UNIV-complete A $(\sqsubseteq)$" **and** "f ` $A \subseteq A$" **and** "monotone_on A $(\sqsubseteq)$ $(\sqsubseteq)$ f"
  shows "UNIV-complete $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$"

[Knaster-Tarski'55] without transitivity and reflexivity
[Stouti-Maaden'13] + completeness without reflexivity

Instances: [Markowsky'76]

theorem (**in** antisymmetric) mono_imp_fp_connex_complete:
  assumes "$\{X.$ connex X $(\sqsubseteq)\}$-complete A $(\sqsubseteq)$"
    and "f ` $A \subseteq A$" **and** "monotone_on A $(\sqsubseteq)$ $(\sqsubseteq)$ f"
  shows "$\{X.$ connex X $(\sqsubseteq)\}$-complete $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$"

[Markowsky'76] without transitivity and reflexivity

Instances: [Pataraia'97]

```
theorem (in antisymmetric) mono_imp_fp_directed_complete:
  assumes "{X. directed X (⊑)}-complete A (⊑)"
    and "f ` A ⊆ A" and "monotone_on A (⊑) (⊑) f"
  shows "{X. directed X (⊑)}-complete {p ∈ A. f p = p} (⊑)"
```

[Pataraia'97] without transitivity and reflexivity

Instances: [Bhatta-George'11]

theorem (**in** antisymmetric) mono_imp_fp_well_complete:
  assumes "well_complete A ($\sqsubseteq$)" **and** "f \` $A \subseteq A$" **and** "monotone_on A ($\sqsubseteq$) ($\sqsubseteq$) f"
  shows "well_complete $\{p \in A.\ f\ p = p\}$ ($\sqsubseteq$)"

[Bhatta-George'11] without reflexivity and ordinals

What's next?

- Continue developing this archive (towards domain theory?)
- Use this archive (CPS, TRS?, others?)
- Use our "experience" in formalising mathematics in Isabelle to develop other theories (topology, metric spaces, …)