Directed homology

PRIX DOCTORANTS ED STIC 2016

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My submission

- J. Dubut, É. Goubault and J. Goubault-Larrecq. Natural Homology. *In* ICALP'15, LNCS 9135, pages 171-183. Springer, 2015.

- J. Dubut, É. Goubault and J. Goubault-Larrecq. Directed homology theories and Eilenberg-Steenrod axioms. *Applied Categorical Structures*, 2016. To appear.

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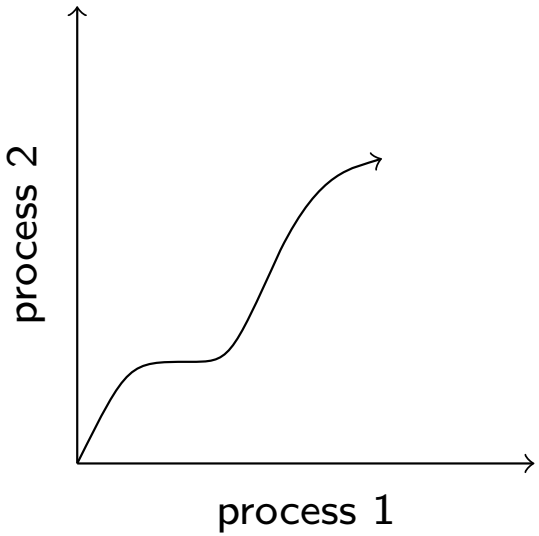
- J. Dubut, É. Goubault and J. Goubault-Larrecq. The Directed Homotopy Hypothesis. *In* CSL'16, Leibniz International Proceedings in Informatics 62, pages 9 :1-9 :16. Leibniz-Zentrum für Informatik, 2016.

I.

From geometric models to directed homology

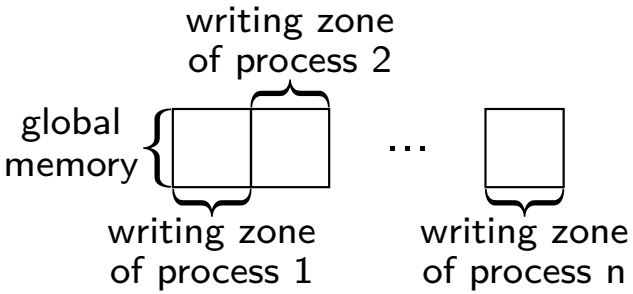
Models for true concurrency

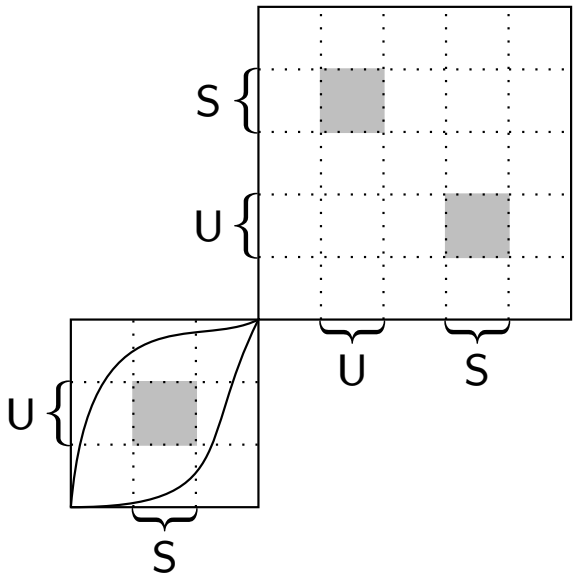


- Petri nets **[Petri 62]**
- progress graphs **[Dijkstra 68]**
- trace theory **[Mazurkiewicz 70s]**
- event structures **[Winskel 80s]**
- geometric model of true concurrency : higher dimension automata (HDA) **[Pratt 91]**

A toy language : SU-programs **[Afek et al. 90]**

- global shared memory
- atomic operations :
  - S : scan the whole memory
  - U : update its own part of the memory
- synchronization $\bullet$ (rendez-vous)
- S are U non independent





$$(S|U) \bullet (U.S|U.S)$$

Objective

Objective of directed algebraic topology :

Compare spaces **with a notion of order** up to continuous deformation **that preserves this order**

Problem coming from :

- geometric semantics of truly concurrent systems
  - PV-programs **[Dijkstra 68]**
  - SU-programs **[Afek et al. 90]**
  - higher dimensional automata **[Pratt 91]**
- theory of relativity **[Dodson, Poston 97]**

Non directed case : algebraic topology

Non directed case : algebraic topology

Compare spaces ~~with a notion of order~~ up to continuous deformation ~~that preserves this order~~

Homology **[Poincaré 1895]** which is :

- sound (invariant of homotopy)
- partially complete **[Hurewicz 52, Whitehead 49]**
- computable **[Poincaré 1900]**
- modular (homology can be expressed from homology of simpler spaces **[Mayer, Vietoris 30]**)

Define a directed analogue of homology

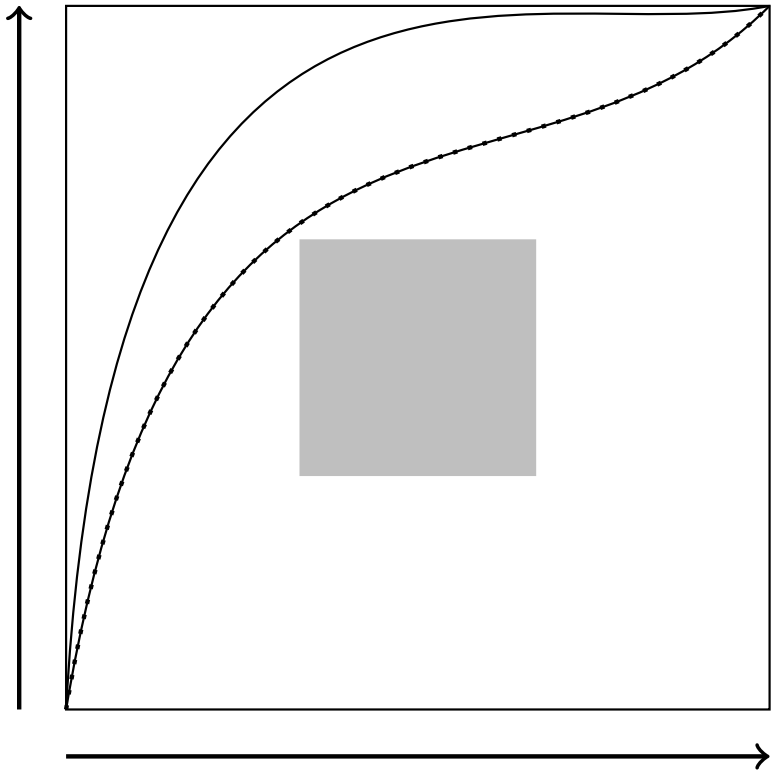
II.

Directed homologies

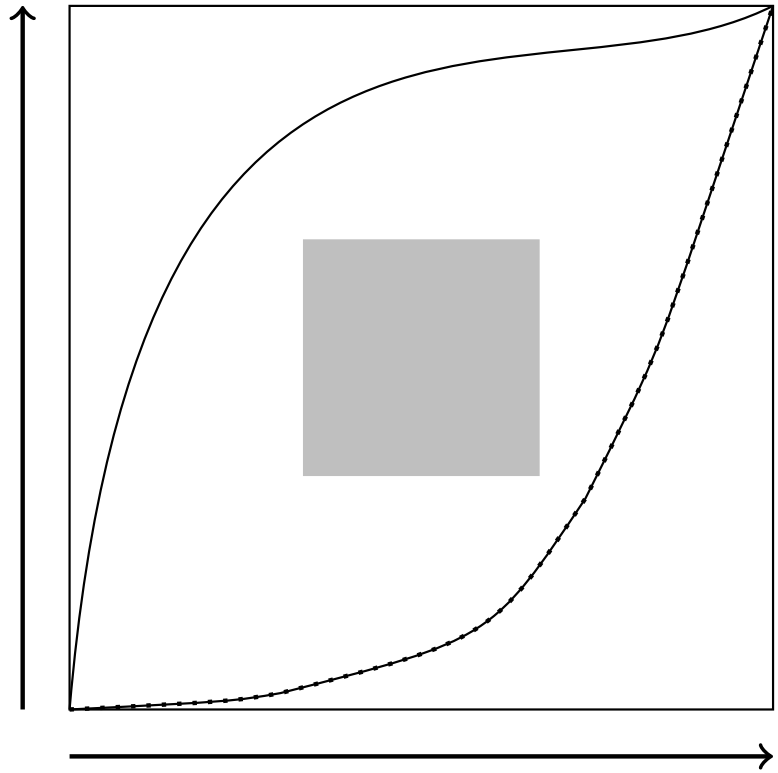
Dihomotopies

Dipaths = **increasing** continuous functions from $[0, 1]$ to X

2 dipaths are **di**homotopic = you can deform continuously one into the other **while staying a dipath**

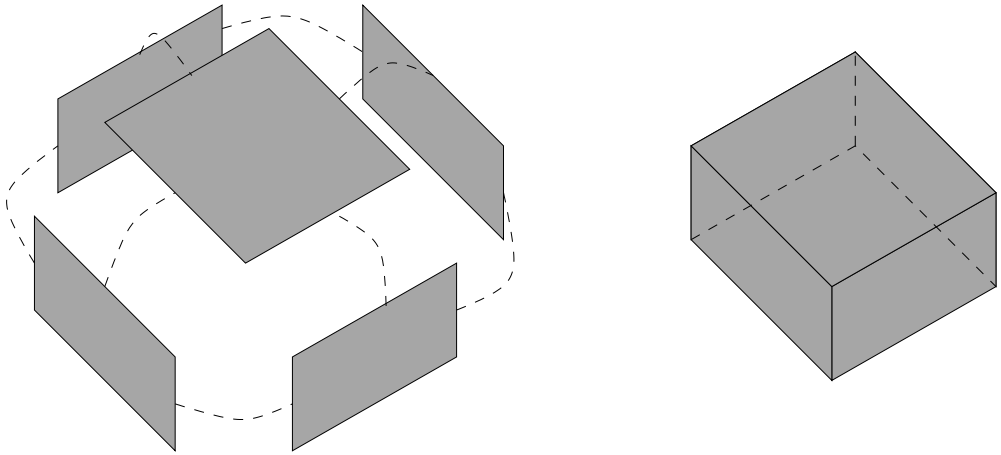


(di)homotopic



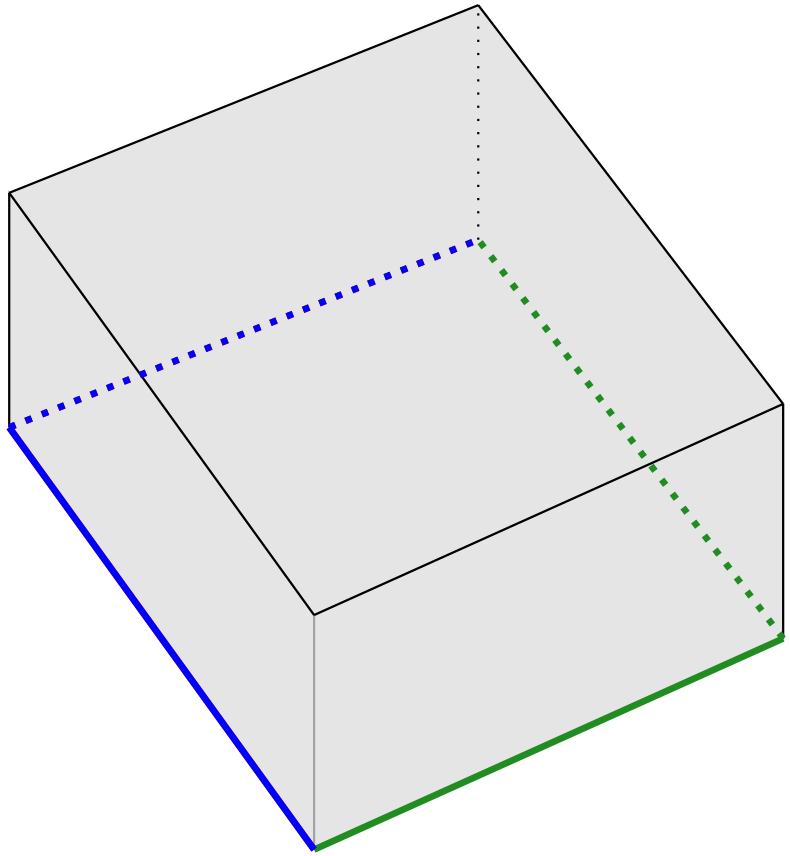
non (di)homotopic

Homotopy vs dihomotopy



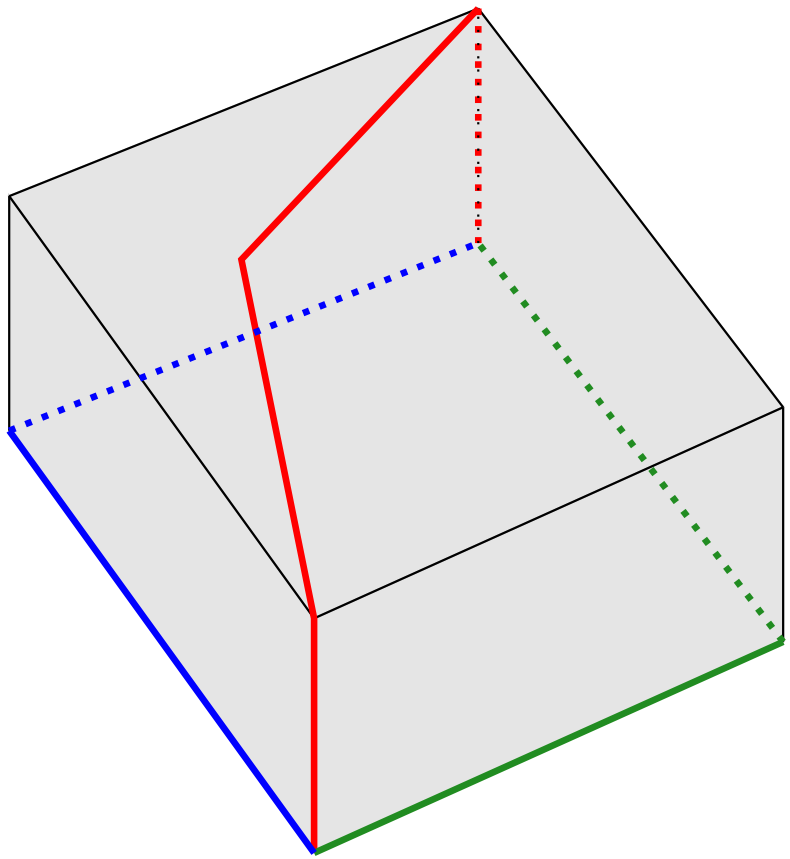
Fahrenberg's matchbox **[Fahrenberg 04]**

Homotopy vs dihomotopy



homotopic...

Homotopy vs dihomotopy



... but not dihomotopic

Related works

Candidates of directed homology :

- past and future homologies **[Goubault 95]**
- ordered homology groups **[Grandis 04]**
- directed homology via ω-categories **[Fahrenberg 04]**
- homology graph **[Kahl 13]**

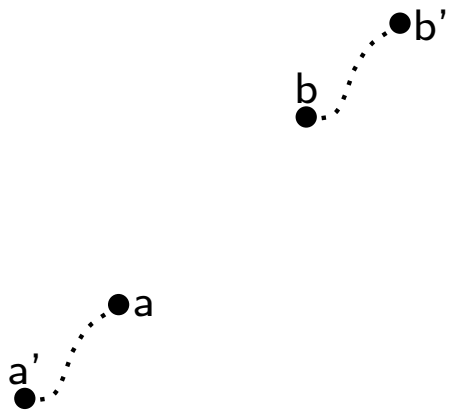
Not fine enough : do not distinguish Fahrenberg's matchbox from a point

III.

Natural homology

Evolution of execution spaces with time

- look at the topological space of dipaths (executions) between pairs of points (states) ;
- consider their classical homology groups ;
- study the evolution of those groups with dynamics.



Natural homology = diagram of those groups and group homomorphisms.

Properties of the natural homology

- soundness
- completeness
- computability
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- modularity
  - $\simeq$ algebraic theory (exactness) **[APCS'16]**

Conclusion

Many questions remains :

- more algebraic topology-like properties ;
- applications in computer science (true concurrency, rewriting, topological data analysis ?), in physics.