Bisimilarity of diagrams
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Jérémy Dubut

National Institute of Informatics, Tokyo, Japan
Japanese-French Laboratory for Informatics, IRL 3527, Tokyo, Japan

Bisimilarity via open morphisms

Computing systems in the language of category theory

Mainly, two types:

- coalgebraic approach **[Rutten, Jacobs, ...]**
- lifting approach **[Winskel, Joyal, Nielsen, ...]**

approach	class type	system type	bisimulations
coalgebraic	category + functor (monad)	coalgebra	span of morphisms of coalgebras
lifting	category + sub-category	object	span of morphisms with lifting property w.r.t. the sub-category

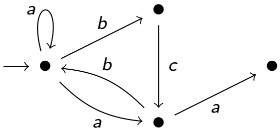
Example : TS I - category of TS

Fix an alphabet Σ.

Transition system :

A **TS** $T = (Q, i, \Delta)$ on Σ is the following data:

- a set Q (of states);
- a initial state $i \in Q$;
- a set of transitions $\Delta \subseteq Q \times \Sigma \times Q$.



Morphism of TS :

A **morphism of TS** $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function $f : Q_1 \longrightarrow Q_2$ such that $f(i_1) = i_2$ and for every $(p, a, q) \in \Delta_1$, $(f(p), a, f(q)) \in \Delta_2$.

TS(Σ) = category of TS on Σ and morphisms of TS

Example : TS II - relational bisimulations

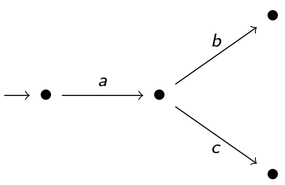
Bisimulations [Park]:

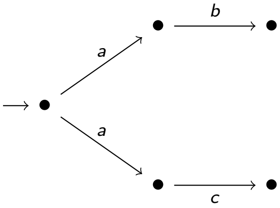
A **bisimulation** between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.





Example : TS III - morphisms and (bi)simulations

$$Graph(f) = \{(q, f(q)) \mid q \in Q\}$$

$Graph(f)$ is always a simulation. But bisimilarity $\neq$ similarity in both directions.

$$\to \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \qquad\qquad \begin{tikzcd} & \bullet \arrow[r, "b"] & \bullet \\ \to \bullet \arrow[ur, "a"] \arrow[dr, "a"'] & & \\ & \bullet & \end{tikzcd}$$

What are the morphisms whose graph is a bisimulation ?

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 & \bullet &
\end{tikzcd}$$

What are the morphisms whose graph is a bisimulation ?

- the morphisms of coalgebras.
- the morphisms that lift transitions.

Example : TS IV - lifting properties and open morphisms

f has the **right lifting property** with respect to g iff

$$
\begin{tikzcd}
P \arrow[r, "x"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Q \arrow[r, "y"'] \arrow[ur, dotted, "\exists\theta"] & Y
\end{tikzcd}
$$

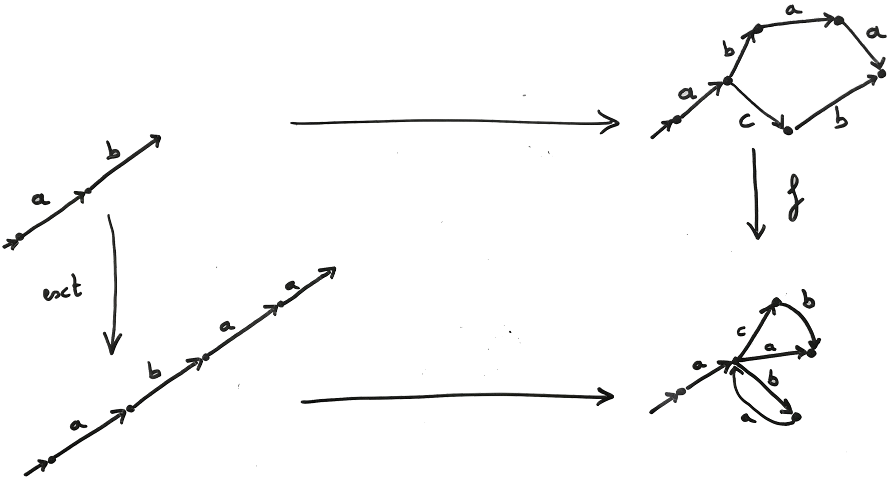
A morphism of TS is **open [Joyal, Nielsen, Winskel]** if it has the right lifting property with respect to every **branch extension**:

$$
\begin{tikzcd}
\arrow[r] & \bullet \arrow[r, "a_1"] \arrow[d, dotted] & \bullet \arrow[d, dotted] & \cdots & \bullet \arrow[r, "a_n"] \arrow[d, dotted] & \bullet \arrow[d, dotted] & & & & \\
\arrow[r] & \bullet \arrow[r, "a_1"] & \bullet & \cdots & \bullet \arrow[r, "a_n"] & \bullet \arrow[r, "a_{n+1}"] & \bullet & \cdots & \bullet \arrow[r, "a_{n+p}"] & \bullet
\end{tikzcd}
$$

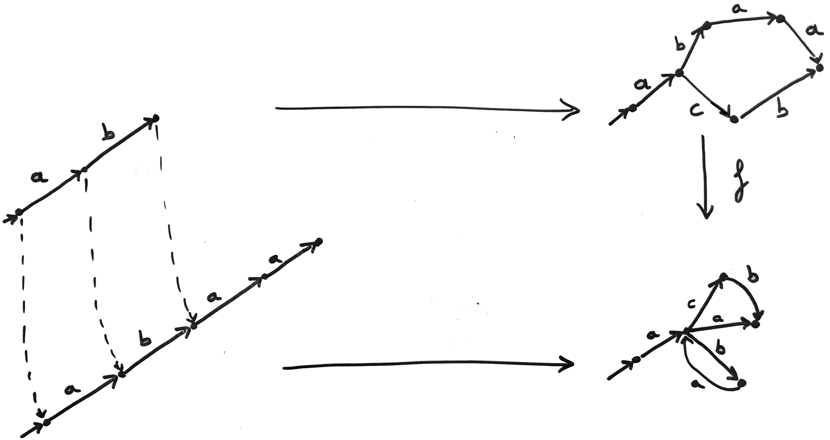
Observation:

Two systems are bisimilar iff there is a span of open morphisms between them.

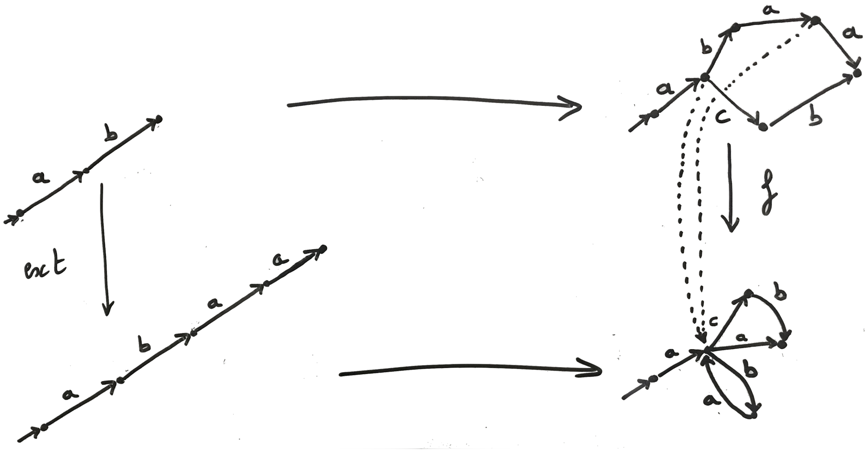
Example of a RLP



Example of a RLP



Example of a RLP



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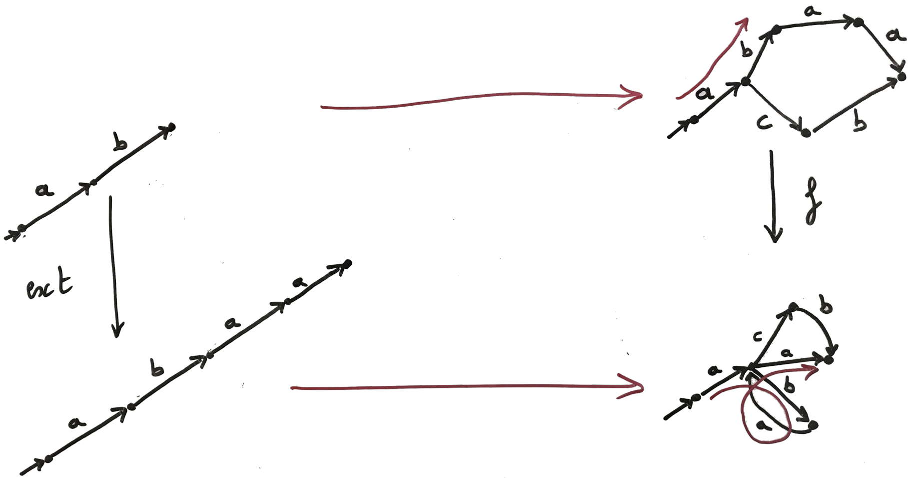
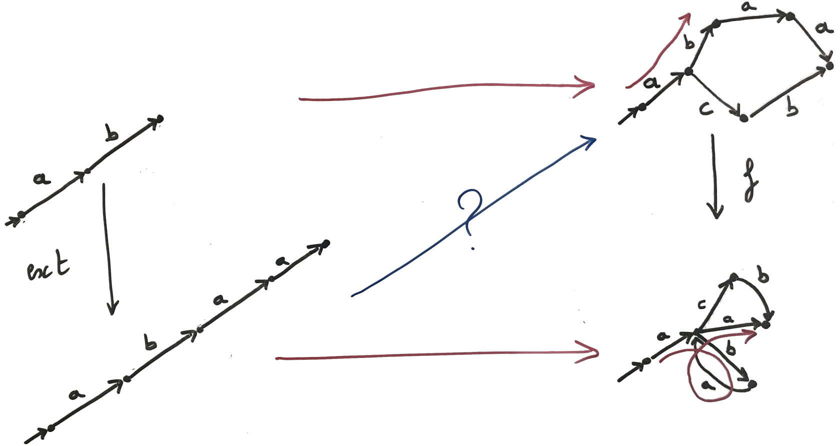
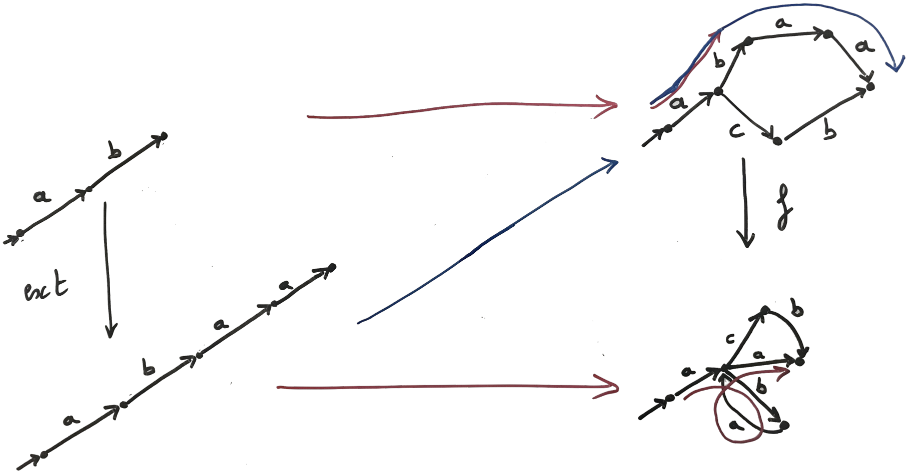


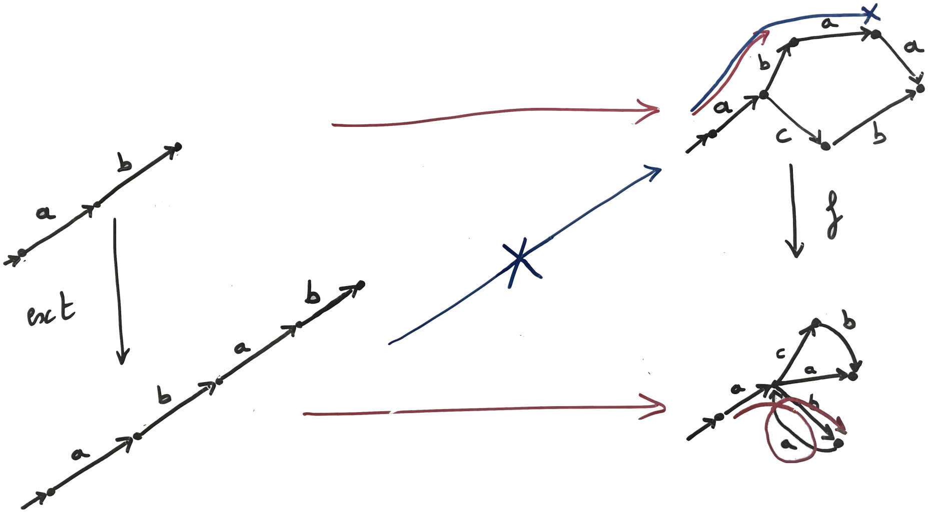
Example of a RLP



Example of a RLP



…but not an open map



Categorical models

Categorical models:

A **categorical model** is a category $\mathcal{M}$ with a subcategory $\mathcal{P}$ which have a common initial object I.

- $\mathcal{M}$ = category of systems (Ex : $\mathbf{TS(\Sigma)}$);
- $\mathcal{P}$ = sub-category of **paths** (Ex : sub-category of branches);
- unique morphism $I \longrightarrow X$ = initial state of X (Ex : $I = *$).

Other examples : 1-safe Petri nets + event structures, event structures/transition systems with indenpendence + pomsets, HDA + paths, presheaves models, ...

Bisimilarity as spans of open morphisms

$\mathcal{P}$-bisimilarity **[J., N., W.]**:

We say that a morphism $f : X \longrightarrow Y$ of $\mathcal{M}$ is **($\mathcal{P}$-)open** if it has the right lifting property w.r.t. $\mathcal{P}$.

$$\begin{tikzcd} P \arrow[r, "x"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ Q \arrow[r, "y"'] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

We then say that two objects X and Y of $\mathcal{M}$ are $\mathcal{P}$-**bisimilar** iff there exists a span $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ where f and g are $\mathcal{P}$-open.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

Ex: strong history-preserving bisimilarity of ES/TSI, ...
Typically, bisimilarity defined by relation on runs.

Path bisimulations

Example : TS V - from states to runs

A bisimulation R between T_1 and T_2 induces a relation R_n between n-branches of T_1 and n-branches of T_2 by:

$$R_n = \{(f_1 : B \longrightarrow T_1, f_2 : B \longrightarrow T_2) \mid \forall i \in [n],\, (f_1(i), f_2(i)) \in R\}$$

Properties:

- $(\iota_{T_1}, \iota_{T_2}) \in R_0$ by (i);
- by (ii), if $(f_1, f_2) \in R_n$ and if $(f_1(n), a, q_1) \in \Delta_1$ then there is $q_2 \in Q_2$ such that $(f_2(n), a, q_2) \in \Delta_2$ and $(f_1', f_2') \in R_{n+1}$ where $f_i'(j) = f_i(j)$ if $j \leq n$, q_i otherwise;
- symmetrically with (iii);
- if $(f_1, f_2) \in R_{n+1}$ then $(f_1', f_2') \in R_n$ where f_i' is the restriction of f_i to $[n]$.

Fact:

Bisimilarity is equivalent to the existence of such a relation between branches.

Relational bisimilarities in categorical models

Let R be a set of elements of the form $X \xleftarrow{f} P \xrightarrow{g} Y$ with P object of $\mathcal{P}$. Here are some properties that R may satisfy:

(a) $X \xleftarrow{\iota_X} I \xrightarrow{\iota_Y} Y$ belongs to R;

(b) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R then for every morphism $p : P \longrightarrow Q$ in $\mathcal{P}$ and every $f' : Q \longrightarrow X$ such that $f' \circ p = f$ then there exists $g' : Q \longrightarrow Y$ such that $g' \circ p = g$ and $X \xleftarrow{f'} Q \xrightarrow{g'} Y$ belongs to R;

$$\begin{tikzcd} X & P \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p"'] & Y \\ & Q \arrow[ul, "f'"] \arrow[ur, dotted, "g'"'] & \end{tikzcd}$$

(c) symmetrically;

(d) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R and if we have a morphism $p : Q \longrightarrow P \in \mathcal{P}$ then $X \xleftarrow{f \circ p} Q \xrightarrow{g \circ p} Y$ belongs to R.

(Strong) path bisimulation [J., N., W.]

When R satisfies (a–c) (resp. (a–d)), we say that it is a **path-bisimulation** (resp. **strong path-bisimulation**).

Facts

To make real sense, $\mathcal{P}$ is needed to be small. In this case:

- $\mathcal{P}$-bisimilarity $\Rightarrow$ strong path-bisimilarity $\Rightarrow$ path-bisimilarity **[J., N., W.]**.
- In many cases, $\mathcal{P}$-bisimilarity is equivalent to strong path-bisimilarity. There is a general framework ($\mathcal{P}$-accessible categories) where it is the case **[D., Goubault, Goubault]**.
- A Hennessy-Milner-like theorem holds for both (strong) path-bisimilarities **[J., N., W.]**.

Bisimilarity of diagrams, via open maps

Category of diagrams

A **diagram** in a category $\mathcal{A}$ is a functor F from any small category $\mathcal{C}$ to $\mathcal{A}$

My view:

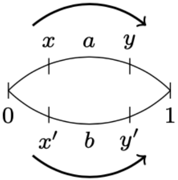
- $\mathcal{C}$ = category of runs,
- $\mathcal{A}$ = category of values (ex: words),
- F = describe the data of each run and how those data evolve (ex: labelling).

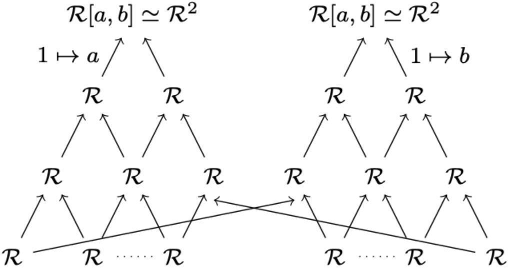
A **morphism of diagrams** from $F : \mathcal{C} \longrightarrow \mathcal{A}$ to $G : \mathcal{D} \longrightarrow \mathcal{A}$ is a pair (Φ, σ) of:

- a functor $\Phi : \mathcal{C} \longrightarrow \mathcal{D}$,
- a natural *isomorphism* $\sigma : F \Longrightarrow G \circ \Phi$.

We note $\mathbf{Diag(}\mathcal{A}\mathbf{)}$ this category.

Where is it from?





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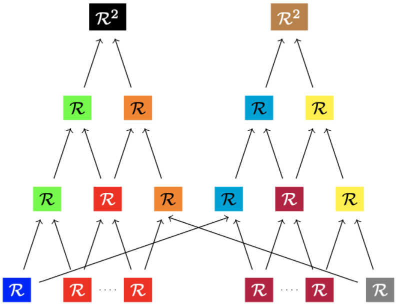

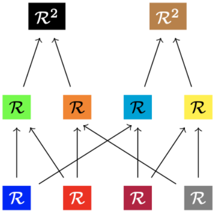

Example : TS VI - from TS to diagrams

T a TS on Σ.

- $\mathcal{C}_T$ = poset of runs, with the prefix order,
- $\mathcal{A}$ = poset $\Sigma^\star$, with the prefix order,
- F_T = maps a run on its labelling.

From small categorical model to diagrams

$\mathcal{M}$ a categorical model, with a small subcategory $\mathcal{P}$
X object of $\mathcal{M}$.

- $\mathcal{C}_X = \mathcal{P} \downarrow X$, whose objects are morphisms in $\mathcal{M}$ from an object of $\mathcal{P}$ to X,
- $\mathcal{A} = \mathcal{P}$,
- $F_X =$ projection on the domain of the morphism.

Remarks:

- This defines a functor Π from $\mathcal{M}$ to $\mathbf{Diag(}\mathcal{P}\mathbf{)}$.
- When $\mathcal{M}$ is cocomplete, the colimit functor Γ from $\mathbf{Diag(}\mathcal{P}\mathbf{)}$ to $\mathcal{M}$ is the left adjoint of Π and $\Gamma \circ \Pi$ is the unfolding.
- The counit $\epsilon_X : \Gamma \circ \Pi(X) \longrightarrow X$ is an open morphisms in many cases.

Lifting properties and open morphisms (in **Diag($\mathcal{A}$)**)

f has the **right lifting property** with respect to g iff

$$\begin{tikzcd} P \arrow[r, "x"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "y"'] \arrow[ur, dotted, "\exists\theta"] & Y \end{tikzcd}$$

A morphism of diagrams is **open** if it has the right lifting property with respect to every **branch extension** ($n \geq 0$):

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, dotted, "\mathrm{id}"'] & A_2 \arrow[d, dotted, "\mathrm{id}"'] & \cdots A_{n-1} \arrow[r, "f_n"] \arrow[d, dotted, "\mathrm{id}"'] & A_n \arrow[d, dotted, "\mathrm{id}"'] & & \\ A_1 \arrow[r, "f_1"'] & A_2 & \cdots A_{n-1} \arrow[r, "f_n"'] & A_n \arrow[r, "f_{n+1}"'] & A_{n+1} \cdots A_{n+p-1} \arrow[r, "f_{n+p}"'] & A_{n+p} \end{tikzcd}$$

Definition:

Two diagrams are bisimilar iff there is a span of open morphisms between them.

Open maps of systems vs. open maps of diagrams

Remember the adjunction:

$$\Gamma : \mathbf{Diag}(\mathcal{P}) \longrightarrow \mathcal{M}$$
$$\perp$$
$$\Pi : \mathcal{M} \longrightarrow \mathbf{Diag}(\mathcal{P})$$

Proposition [D.]:

If $f : X \longrightarrow Y$ is an open morphism of systems, then $\Pi(f) : \Pi(X) \longrightarrow \Pi(Y)$ is an open morphism of diagrams. In particular, if X and Y are bisimilar, then $\Pi(X)$ and $\Pi(X)$ are bisimilar.

The converse is not true in general.

For example, that is not true in general that if $\Pi(X) \xleftarrow{\Phi} Z \xrightarrow{\Psi} \Pi(Y)$ is a span of open morphisms then $\Gamma \circ \Pi(X) \xleftarrow{\Gamma(\Phi)} \Gamma(Z) \xrightarrow{\Gamma(\Psi)} \Gamma \circ \Pi(Y)$ is a span of open morphisms.

Bisimilarity of diagrams, via bisimulations

Bisimulation of diagrams

Bisimulation between $F : \mathcal{C} \longrightarrow \mathcal{A}$ and $G : \mathcal{D} \longrightarrow \mathcal{A}$
= set R of triples (c, η, d) such that :

- c is an object of $\mathcal{C}$,
- d is an object of $\mathcal{D}$,
- $\eta : F(c) \longrightarrow G(d)$ is an isomorphism of $\mathcal{A}$

satisfying :

- for every object c of $\mathcal{C}$, there exists d and η such that $(c, \eta, d) \in R$
-

$$(c, \eta, d) \in R$$

$$\begin{tikzcd}
c \arrow[d, "i"'] & Fc \arrow[r, "\eta"] \arrow[d, "Fi"'] & Gd \arrow[d, dotted, "Gj"] & d \arrow[d, dotted, "j"] \\
c' & Fc' \arrow[r, dotted, "\eta'"'] & Gd' & d'
\end{tikzcd}$$

$$(c', \eta', d') \in R$$

and symmetrically

Bisimilarity and bisimulations

Theorem [D.]:

Two diagrams are bisimilar if and only if there is a bisimulation between them.

Proof sketch:

$\Rightarrow$ Given a span $F \xleftarrow{(\Phi,\sigma)} (H : \mathcal{E} \longrightarrow \mathcal{A}) \xrightarrow{(\Psi,\tau)} G$ of open maps:

$$\{(\Phi(e), \tau_e \circ \sigma_e^{-1}, \Psi(e)) \mid e \in Ob(\mathcal{E})\}$$

$\Leftarrow$ Given a bisimulation R, construct a diagram H:

- whose domain is R,
- which maps (c, η, d) to $F(c)$.

The projections from H to F and G are open.

A word on (un)decidability

Bisimulation = relation + isomorphisms

In a finite case: guess the relation $\Rightarrow$ problem of isomorphisms in $\mathcal{A}$.

For example, in a vector spaces, we are left with this problem:
Data: a set of equations in matrices of the $X.A = B.Y$
Question: are there invertible matrices X, Y, ... that satisfy the equations ?

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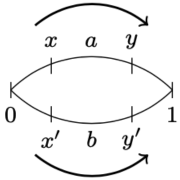

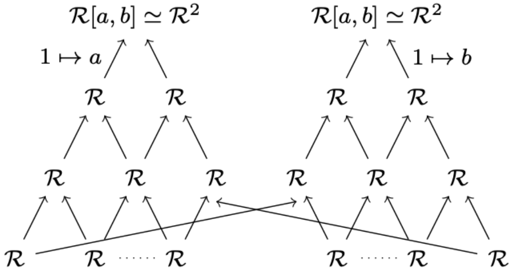

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Proposition **[D.]**:

In a finite case, bisimilarity is:

- decidable if $\mathcal{A}$ is finite or **FinSet**,
  - a finite number of possible solutions
- undecidable if $\mathcal{A}$ = category of finitely presented groups + group morphisms,
  - isomorphism is undecidable
- decidable if $\mathcal{A}$ = category of finite dimensional real (rational) vector spaces,
  - can be reduced to polynomial equations in reals
- open if $\mathcal{A}$ = category of Abelian groups of finite type.

Bisimulations of systems vs. bisimulations of diagrams

Remember the adjunction, again:

$$\Gamma : \mathbf{Diag}(\mathcal{P}) \longrightarrow \mathcal{M}$$
$$\perp$$
$$\Pi : \mathcal{M} \longrightarrow \mathbf{Diag}(\mathcal{P})$$

Proposition **[D.]**:

Bisimulations of diagrams between $\Pi(X)$ and $\Pi(Y)$ are precisely path-bisimulations between X and Y.

Corollary:

$\Pi(X)$ and $\Pi(Y)$ are bisimilar iff X and Y are path-bisimilar.

Remarks:

- This explains why open in systems $\neq$ open in diagrams.
- For transition systems, this implies that two systems are bisimilar iff the diagrams are bisimilar.

What about strong path-bisimulations ?

What is missing ?

Being able to reverse paths.

Solution: just add reverse of paths !

Given a category $\mathcal{C}$, define $\overline{\mathcal{C}}$ as the category generated by $\mathcal{C} \cup \mathcal{C}^{op}$, i.e.:

- objects are those of $\mathcal{C}$,
- morphisms are zigzags of morphisms of $\mathcal{C}$

$$\begin{tikzcd}
 & c_1 & & c_3 & & & c_{n-1} & \\
c_0 \arrow[ur, "f_1"] & & c_2 \arrow[ul, "f_2"'] \arrow[ur, "f_3"] & & \dots & c_{n-2} \arrow[ur, "f_{n-1}"] & & c_n \arrow[ul, "f_n"']
\end{tikzcd}$$

A functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ induces a functor $\overline{F} : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{A}}$ and this extends to a functor $\Delta : \mathbf{Diag}(\mathcal{A}) \longrightarrow \mathbf{Diag}(\overline{\mathcal{A}})$.

Instead of looking at $\Pi(X)$, we look at $\Delta \circ \Pi(X)$.

Bisimulations of systems vs. bisimulations of diagrams II

We still have a adjunction:

$$\begin{array}{c} \Gamma' : \mathbf{Diag}(\overline{\mathcal{P}}) \longrightarrow \mathcal{M} \\ \perp \\ \Delta \circ \Pi : \mathcal{M} \longrightarrow \mathbf{Diag}(\overline{\mathcal{P}}) \end{array}$$

Proposition [D.]:

Bisimulations of diagrams between $\Delta \circ \Pi(X)$ and $\Delta \circ \Pi(Y)$ are precisely strong path-bisimulations between X and Y.

Corollary:

$\Delta \circ \Pi(X)$ and $\Delta \circ \Pi(Y)$ are bisimilar iff X and Y are strong path-bisimilar.

Remarks:

- In many cases, $\Delta \circ \Pi(X)$ and $\Delta \circ \Pi(Y)$ are bisimilar iff X and Y are $\mathcal{P}$-bisimilar.
- In the $\mathcal{P}$-accessible case, Γ' maps open maps to open maps.

Conclusion

Conclusion

We have a theory of bisimilarity of diagrams:

- defined using open maps,
- equivalent characterization using bisimulations,
- decidability is essentially a problem of isomorphism in the category of values,
- models (strong) path-bisimilarities,
- useful in directed algebraic topology (not too much in this talk),
- admits a Hennessy-Milner-like theorem (not in this talk).

What is left (inter alia):

- open morphisms acts like trivial fibrations. Can we make that explicit ?
- (un)decidability in the case of Abelian groups.
- relation between our decision procedure to usual ones.