Natural Homology

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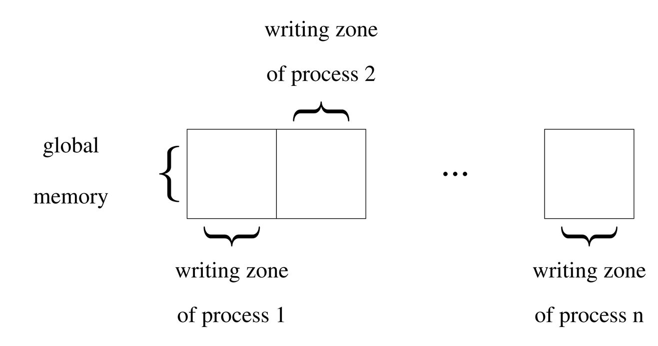
Objectives

We want to define a suitable notion of **homology for directed algebraic topology** i.e. on spaces with a notion of **order/directed paths**. This problematic was born out of the so-called geometric semantics of truly concurrent processes. Imagine n concurrent processes, each with a local time $t_i \in [0,1]$. A configuration is a point in $[0,1]^n$, and a trajectory is a continuous **and monotonic** map from $[0,1]$ to $[0,1]^n$: monotonicity (a.k.a., directedness) reflects the fact that no process can go back in time. One can arguably consider as equivalent any two trajectories that are dihomotopic, namely that can be deformed into each other continuously, while respecting **monotonicity** at all times. For example, the SU-programs can be realized as in figure 1:

- a shared global memory
- 2 atomic operations:
  - S : scan all the memory
  - U : update its own part of the memory
- synchronisation • (rendez-vous)
- S and U non independent



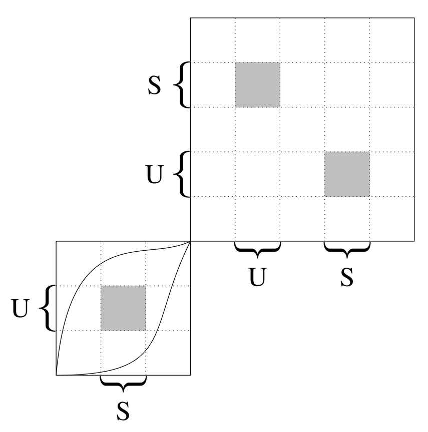


Figure 1: geometric realization of the SU-program $(S|U) \bullet (U.S|U.S)$

Homology is a classical concept in (undirected) algebraic topology. Many attempts to define a directed analogue have the same weakness: they are not precise enough. By this we mean, that directed homology should not be invariant under (undirected) homotopy. If it were, it would be blind to the essential feature of directed algebraic topology: **that directions are important**. So, we would like that any dihomotopically non-trivial shape to have non-trivial directed homology (for example, a Hurewicz-like theorem). This fails in many proposals. Consider the *matchbox* example.

As a topological space, it is contractible and so proposals that define dihomology as classical homology with extra structure are trivial on this space. In particular, the blue path is homotopic to the green one, through the red one for example. But as a directed space, it is dihomotopically non-trivial. Indeed, the blue and green dipaths are not dihomotopic, because to deform one into the other we must go through a non-directed path (for example the red one).

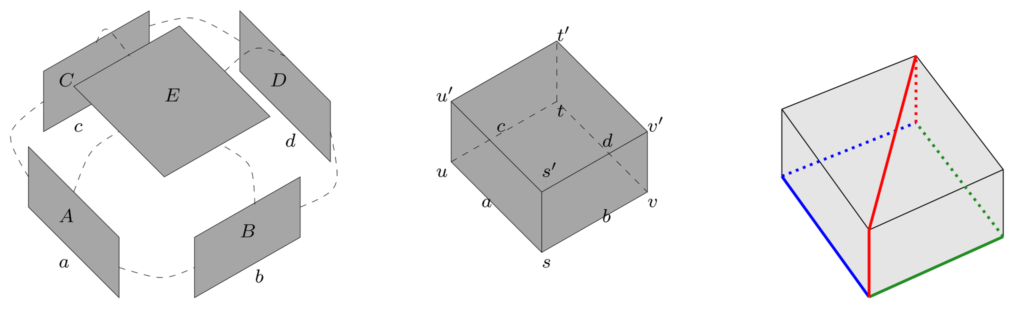


Figure 2: Fahrenberg's matchbox

Objectives : define a dihomology

- **which is non-trivial on dihomotopically non-trivial spaces**
- **whose informations are computable on finitely presented spaces**
- **which has some nice theoretical properties (Hurewicz-like, long exact sequences, ...)**
- **which is invariant at least by dihomeomorphism and subdivision but not by homotopy equivalence**

Natural systems of homology

For the geometric realization of a SU-program, a first natural definition of a dihomology could be the classical homology of the space of traces (i.e. directed paths modulo directed reparametrization) between the initial state to the end state. The idea is that n-directed loops are $(n-1)$-loops of a space of traces [6]. However, that is not sufficient to classify programs ! Let us consider these two SU-programs:

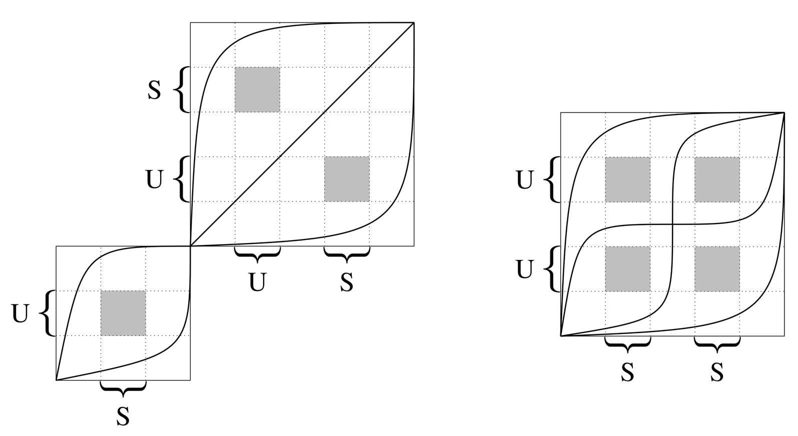
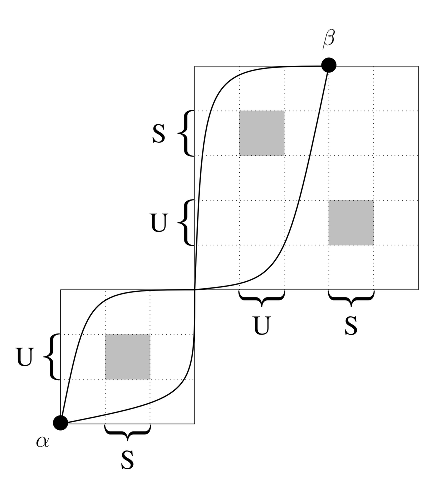


Figure 3: $(S|U) \bullet (U.S|U.S)$ and $S.S|U.U$

However, that is not sufficient to classify programs ! Let us consider the two SU-programs on the left. Their trace spaces from their bottom-left point to their top-right point are homotopically equivalent to a 6 point space (i.e. in those two programs, they are 6 equivalence classes of total executions). And so, this first definition of dihomology does not distinguish these programs that have very different behaviors. Our idea is to **replace this homology of a trace space by a natural system [1] of homologies of all trace spaces between two accessible points and the way they vary when we extend the traces**.

It will be sufficient to distinguish the two SU-programs. Indeed, in the left one, the trace space between α and β is homotopically equivalent to a 4 point space, but there is no pair of points in the right one between which the trace space is of this homotopy type. Thus in the first homology system of the left program, there will be a group isomorphic to $\mathbb{Z}^4$ but not in the one of the right program. Also, the first homology system of the matchbox is not trivial because the trace space between s and t is homotopically equivalent to a 2 point space.



More concretely, the nth homology system of a directed space will be the functor defined this way [6]:

trace p p dipath from a to b	$\longmapsto$	H_{n-1}(space of traces from a to b)
extension (α,β) α from a' to a, β from b to b'	$\longmapsto$	morphism induced in homology by concatenation with α on the left and β on the right

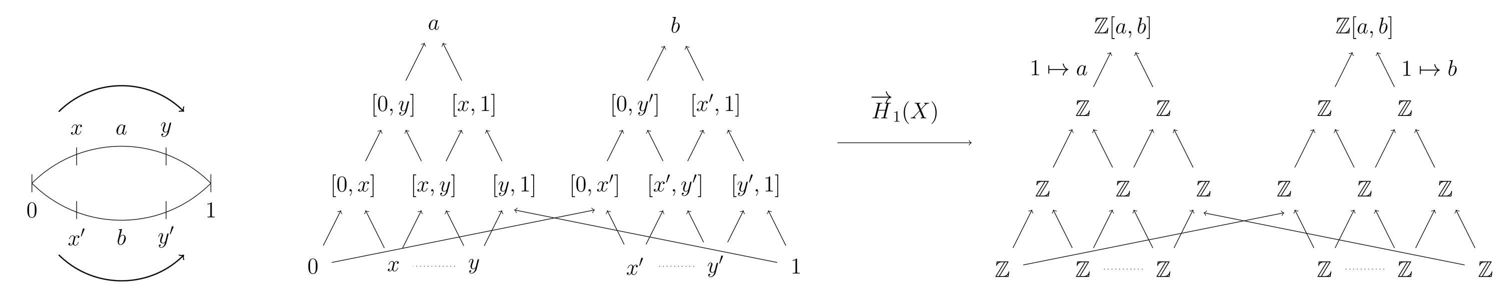


Figure 4: example of a first homology system

Equivalence of systems

The natural homology of a directed space is incredibly **fine-grained**: it not only records local homology groups of all the trace spaces but also for which trace they occur. If we wish to compare the natural homology of two directed spaces, the latter should be unimportant. It is the **patterns of change** when we extend traces that count, not the value at each trace. We have introduced a notion of bisimulation of natural systems that smoothes this out [2].

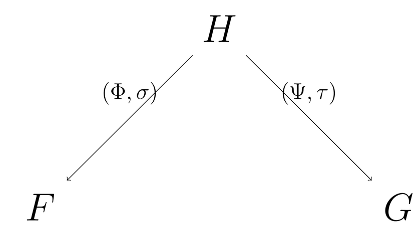


Figure 5: span of open maps

This comes from the theory of open maps [5]. In our case, an open map between small **Ab**-valued functors $F : X \longrightarrow$ **Ab** and $G : Y \longrightarrow$ **Ab** is a pair of:

- a fibration $\Phi : X \longrightarrow Y$ i.e. a functor such that:
  - Φ is surjective on objects
  - for every object x of X, every morphism $f : \Phi(x) \longrightarrow y$ of Y, there exists a morphism $g : x \longrightarrow x'$ of X such that $\Phi(g) = f$
- a natural isomorphism $\sigma : F \longrightarrow G \circ \Phi$

We say that two natural systems F and G are bisimilar if there exists a span of open maps between them.

Discrete systems of homology

When X is the geometric realization of a non-looping precubical set, we can define discrete natural systems that intuitively will have the same information than the natural homology systems of X and that will be finite when the precubical set is. This will be done by considering a sub-category of the category of traces, restricted to some combinatorial traces. With each point of the geometric realization, we associate a cube of the precubical set as the biggest one to which this point belongs. This is the **carrier** of the point [3]. Then with each trace, we can associate the sequence of cubes crossed by this trace and from this sequence of cubes we can construct a trace by joining the centers of consecutive cubes. We can then restrict our natural homology systems to those traces and call these new systems **discrete** natural homology systems.

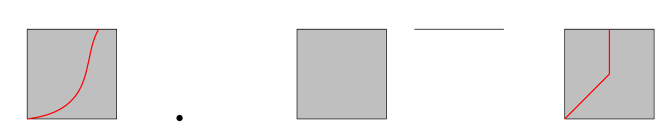


Figure 6: trace, sequence of carriers, combinatorial trace

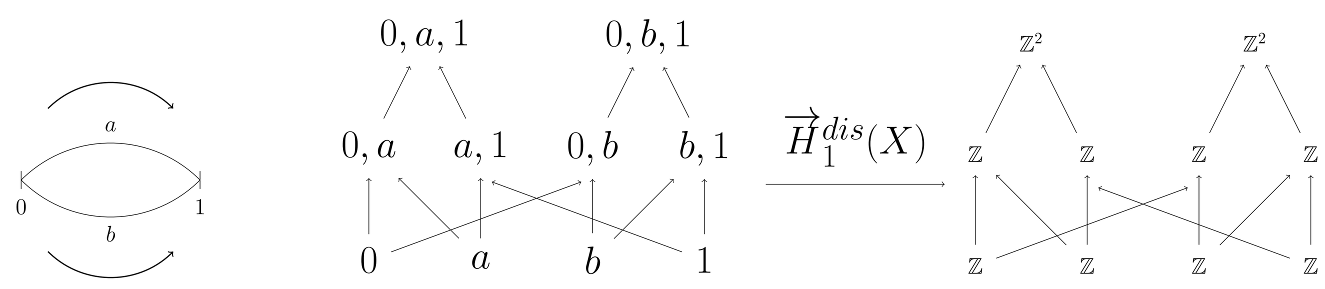


Figure 7: example of a first discrete natural homology system

The function that maps each trace to the combinatorial trace constructed above can always be extended to a fibration, but in general we cannot construct an open map between the natural homology systems and the discrete one. But it can be done in cubical complexes:

Theorem [2]: If X is the geometric realization of a cubical complex, then
- there exists an open map from $\overrightarrow{H}_n(X)$ to $\overrightarrow{H}^{dis}_n(X)$ (in particular, they are bisimilar).
- the bisimulation type of discrete natural homology systems is invariant under subdivision
- if the cubical complex is finite, the bisimulation type of $\overrightarrow{H}_n(X)$ is computable when homology is taken in $\mathbb{R}$ or $\mathbb{Q}$.

Conclusion and references

Definition of a dihomology which :

- **is computable on finite cubical complexes**
- **classifies the matchbox correctly**
- **is invariant under dihomeomorphism and subdivision**
- **has long exact sequences (natural homology lives in a homological category [4])**
- **verifies a Hurewicz-like theorem**

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