Weighted and Branching Bisimilarities from Generalized Open Maps

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Categorical Formalisation of Transition Systems

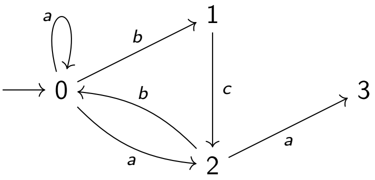
Transition Systems

Labelled transition system :

A **TS** $T = (Q, i, \Delta)$ on the alphabet Σ is the following data:

- a set Q (of **states**);
- an **initial state** $i \in Q$;
- a set of **transitions** $\Delta \subseteq Q \times \Sigma \times Q$.

- $\Sigma = \{a, b, c\}$,
- $Q = \{0, 1, 2, 3\}$,
- $i = 0$,
- $\Delta = \{(0, a, 0), (0, b, 1), (0, a, 2), (1, c, 2), (2, b, 0), (2, a, 3)\}$.



Bisimulations of Transition Systems

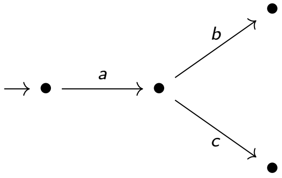
Strong bisimulations [Park81] :

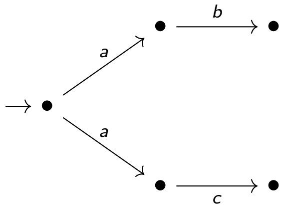
A **bisimulation** between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.





Several Characterisations of Bisimilarity

Bisimilarity:

Given two TS T and T', the following are equivalent:

- **[Park81]** There is a bisimulation between T and T'.
- **[Stirling96]** Defender has a strategy to never lose in a 2-player game on T and T'.
- **[Hennessy80]** T and T' satisfy the same formulas of the Hennessy-Milner logic.

In this case, we say that T and T' are **strongly bisimilar**.

Morphisms of Transition Systems

Morphism of TS:

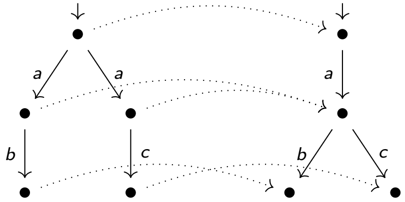
A **morphism of TS** $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function

$$f : Q_1 \longrightarrow Q_2$$

such that:

- **preserving the initial state:** $f(i_1) = i_2$,
- **preserving the transitions:** for every $(p, a, q) \in \Delta_1$, $(f(p), a, f(q)) \in \Delta_2$.

TS(Σ) = category of transition systems and morphisms



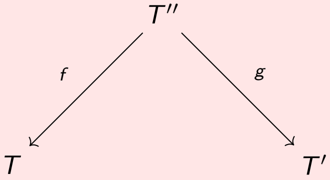
Morphisms are functional simulations:

Morphisms are precisely functions f between states whose graph $\{(q, f(q)) \mid q \in Q_1\}$ is a simulation.

Categorical Characterisations

Bisimilarity, using morphisms:

Two TS T and T' are bisimilar iff there is a span of functional bisimulations between them.



Transition systems, as pointed coalgebras

Set of transitions, as functions:

There is a bijection between sets of transitions $\Delta \subseteq Q \times \Sigma \times Q$ and functions of type:

$$\delta : Q \longrightarrow \mathcal{P}(\Sigma \times Q)$$

where $\mathcal{P}(X)$ is the powerset $\{U \mid U \subseteq X\}$.

Initial states, as functions:

There is a bijection between initial states $i \in Q$ and functions of type:

$$\iota : * \longrightarrow Q$$

where $*$ is a singleton.

Pointed coalgebras

Pointed coalgebras:

Given an endofunctor $G : \mathcal{C} \longrightarrow \mathcal{C}$ and an object $I \in \mathcal{C}$, a **pointed coalgebra** is the following data:

- an object $Q \in \mathcal{C}$,
- a morphism $\iota : I \longrightarrow Q$ of $\mathcal{C}$,
- a morphism $\sigma : Q \longrightarrow G(Q)$ of $\mathcal{C}$.

G is often decomposed as $T \circ F$, where:

- T: "branching type", e.g, non-deterministic, probabilistic, weighted.
  For TS: $T = \mathcal{P}$.
- F: "transition type".
  For TS: $F = \Sigma \times \_$.

I is often the final object

For TS: $I = *$, the final object.

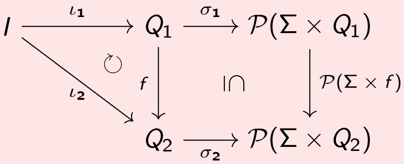
Morphisms of TS, using Pointed Coalgebras

Morphisms of TS are lax morphisms of pointed coalgebras

A morphism of TS, seen as pointed coalgebras $T = (Q_1, \iota_1, \delta_1)$ and $T' = (Q_2, \iota_2, \delta_2)$ is the same as a function

$$f : Q_1 \longrightarrow Q_2$$

satisfying



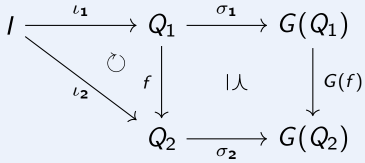
Lax Morphisms of Pointed Coalgebras

Lax Morphisms:

Assume there is an order $\preceq$ on every Hom-set of the form $\mathcal{C}(X, G(Y))$. A **lax morphism** from (Q_1, ι_1, δ_1) to (Q_2, ι_2, δ_2) is a morphism

$$f : Q_1 \longrightarrow Q_2$$

of $\mathcal{C}$ satisfying



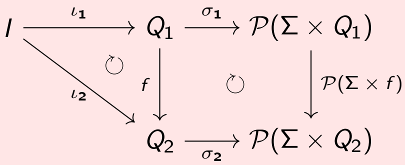
$\mathbf{Coal}_{\mathrm{lax}}(G, I)$ = category of pointed coalgebras and lax morphisms.

What about functional bisimulations?

Functional bisimulations are homomorphisms of pointed coalgebras

For two TS, seen as pointed coalgebras $T = (Q_1, \iota_1, \delta_1)$ and $T' = (Q_2, \iota_2, \delta_2)$, and for a function of the form $f : Q_1 \longrightarrow Q_2$, the following are equivalent:

- The graph $\{(q, f(q)) \mid q \in Q_1\}$ of f is a bisimulation.
- f is a homomorphism of pointed coalgebras, that is, the following diagram commutes:



Bisimilarity, using homomorphisms of pointed coalgebras

For two TS T and T', the following are equivalent:

- T and T' are bisimilar.
- There is a span of homomorphisms of pointed coalgebras between T and T'.

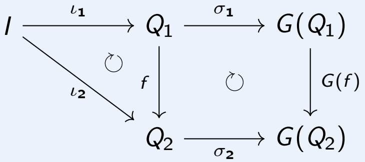
Homomorphisms of Pointed Coalgebras

Morphisms:

A **homomorphism** from (Q_1, ι_1, δ_1) to (Q_2, ι_2, δ_2) is a morphism

$$f : Q_1 \longrightarrow Q_2$$

of $\mathcal{C}$ satisfying



Coal(G, I) = category of pointed coalgebras and homomorphisms.

Summary

	coalgebra	
data type	$G : \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ $\preceq$ on $\mathcal{C}(X, G(Y))$	
systems	pointed coalgebras	
functional simulations	lax morphisms	
functional bisimulations	homomorphisms	
bisimilarity	existence of a span of functional bisimulations	

Runs in a Transition System

Run

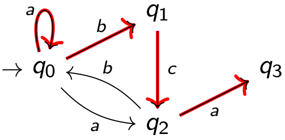
A **run** in a transition system (Q, i, Δ) is sequence written as:

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$$

with:

- $q_j \in Q$ and $a_j \in \Sigma$
- $q_0 = i$
- for every j, $(q_j, a_{j+1}, q_{j+1}) \in \Delta$

$$q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{c} q_2 \xrightarrow{a} q_3$$



Runs, Categorically

Finite Linear Systems:

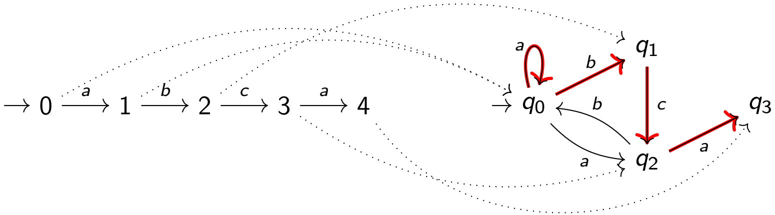
A **finite linear system** is a TS of the form $\langle a_1, \ldots, a_n \rangle = ([n], 0, \Delta)$ where:

- $[n]$ is the set $\{0, \ldots, n\}$;
- Δ is of the form $\{(i, a_{i+1}, i+1) \mid i \in [n-1]\}$ for some a_1, ..., a_n in Σ.

$$\to 0 \xrightarrow{a_1} 1 \xrightarrow{a_2} 2 \quad \cdots \quad n-1 \xrightarrow{a_n} n$$

Runs are morphisms

There is a bijection between runs of T and morphisms of TS between a finite linear system to T.

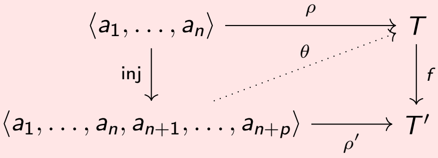


Functional Bisimulations, from Lifting Properties of Paths

Functional bisimulations are open maps:

For a morphism f of TS from T to T', the following are equivalent:

- The reachable graph of f, that is, $\{(q, f(q)) \mid q \ \text{ reachable}\}$ is a bisimulation.
- f has the right lifting property w.r.t. path extensions, that for every commutative square (in plain):



  there is a lifting (in dot), making the two triangles commute.

Bisimilarity, using open maps

For two TS T and T', the following are equivalent:

- T and T' are bisimilar.
- There is a span of open maps between T and T'.

Open maps

Open map situation:

An **open map situation** is a category $\mathbb{M}$ (of **systems**) together with a subcategory $J : \mathbb{P} \hookrightarrow \mathbb{M}$ (of **paths**).

- $\mathbb{M}$ = category of systems (Ex : $\mathbf{TS}(\Sigma)$),
- $\mathbb{P}$ = sub-category of finite linear systems.

Open maps:

A morphism $f : T \longrightarrow T'$ of $\mathbb{M}$ is said to be **open** if for every commutative square (in plain):

$$\begin{array}{ccc} J(P) & \xrightarrow{\quad\rho\quad} & T \\ {\scriptstyle J(p)}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle f} \\ J(Q) & \xrightarrow[\quad\rho'\quad]{} & T' \end{array}$$

where $p : P \longrightarrow Q$ is a morphism of $\mathbb{P}$, there is a lifting (in dot) making the two triangles commute.

Summary

[Wißman, D., Katsumata, Hasuo – FoSSaCS'19]
$\longrightarrow$
Non-deterministic branching

	coalgebra	open maps
data type	$G : \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ $\preceq$ on $\mathcal{C}(X, G(Y))$	$J : \mathbb{P} \hookrightarrow \mathbb{M}$
systems	pointed coalgebras	objects of $\mathbb{M}$
functional simulations	lax morphisms	morphisms of $\mathbb{M}$
functional bisimulations	homomorphisms	open maps
bisimilarity	existence of a span of functional bisimulations	

[Lasota'02]
$\longleftarrow$
Small category of paths

Coalgebra vs open maps

Coalgebras

Pros:

- it is very easy to generalise to other type of branchings (weighted, probabilistic, ...), and other types of state spaces (Stone spaces for Kripke frames, measurable spaces for probabilistic systems, ...)
- there is a rich theory behind.

Con:

- difficult to model any non-local or history-preserving notion of bisimulations (weak bisimulations, timed systems, true concurrency)

Open Maps

Pro:

- it is very easy to model history-preserving bisimulations.

Cons:

- Limited to non-deterministic branching?
- marginally known and used :-(

Contributions

Negative result

There is no open map situation capturing generative probabilistic systems.

Positive results

- There is a *generalised* open map situation capturing generative probabilistic systems.
- There is a generalised open map situation capturing branching bisimulations and weak bisimulations.

Open Maps and Probabilistic Systems

Weighted and Probabilistic Systems as Coalgebras

Given a monoid $(K, +, e)$, define the endofunctor $\mathcal{W}_K$:

$$\begin{array}{llll} \text{sets:} & X & \mapsto & \mathcal{W}_K(X) = \{\mu\colon X \to K \mid \mu^{-1}(K \setminus \{e\}) \text{ is finite}\} \\ \text{maps:} & f\colon X \to Y & \mapsto & \mathcal{W}_K(f)(\mu) = \left(y \in Y \mapsto \sum \{\mu(x) \mid x \in X, f(x) = y\}\right) \end{array}$$

If $(K, \sqsubseteq)$ is a partial order, $\mathbf{Set}(X, \mathcal{W}_K(Y))$ is partially ordered in a pointwise manner.

If $(K, +, e, \sqsubseteq)$ is an ordered monoid, then $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{W}_K, *)$ is a category.

The distribution functor $\mathcal{D}$ (resp. sub-distribution functor $\mathcal{D}_{\leq 1}$) is the sub-functor of $\mathcal{W}_{([0.+\infty),+,0,\leq)}$ restricted to μs with $\sum \{\mu(x) \mid x \in X\} = 1$ (resp. ≤ 1).

Open Maps for Reactive Systems

[Cheng-Nielsen'95]

There is an open map situation that captures Larsen-Skou bisimilarity.

Proof:

- they consider reactive systems, i.e., coalgebras for $(\mathcal{D}(\_) + 1)^A$,
- they define the open map situation in coalgebras for $(\mathcal{D}^{\epsilon}_{\leq 1}(\_) + 1)^A$
  - paths are finite linear systems whose probabilities are infinitesimals
- two proper reactive systems are Larsen-Skou bisimilar iff they is a span of open maps in non-proper systems between them.

Problems:

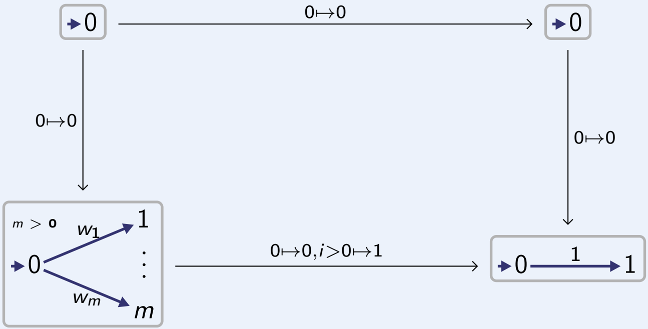
- the open map situation is defined outside of the category of systems,
- Larsen-Skou bisimilarity here means the underlying TS (forgetting the probabilities) are strong bisimilar.

Impossibility for Generative Systems

Theorem:

There is no open maps situation on $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{D}_{\leq 1}(\_ \times A), *)$ such that open bisimilarity coincides with coalgebraic bisimilarity.

Proof



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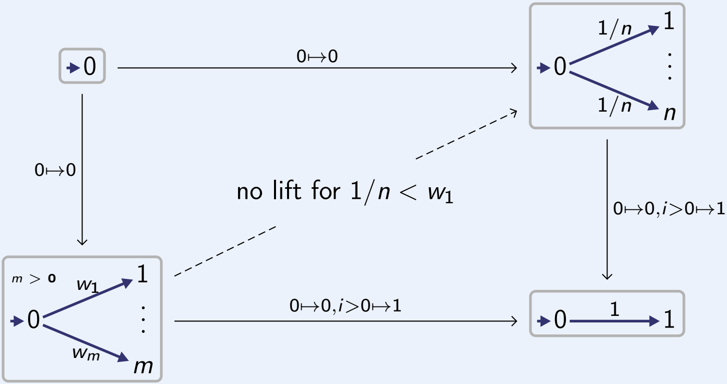
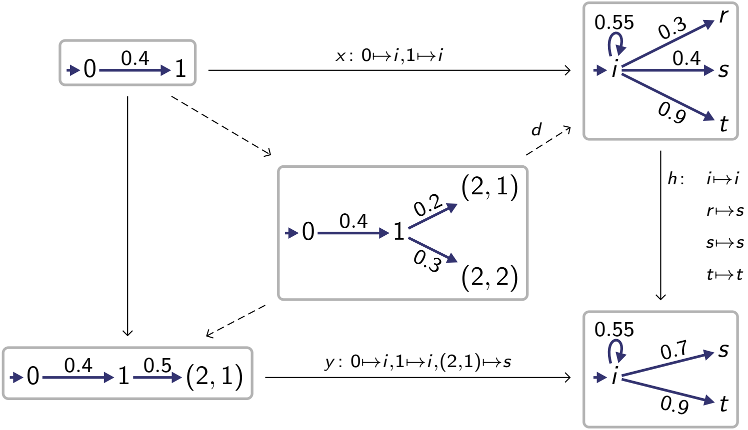
Proof



Solution


0 0.4 1
x: 0↦i,1↦i
0.55 i 0.3 r 0.4 s 0.9 t
d
0 0.4 1 0.2 (2, 1) 0.3 (2, 2)
h: i↦i r↦s s↦s t↦t
0 0.4 1 0.5 (2, 1)
y: 0↦i,1↦i,(2,1)↦s
0.55 i 0.7 s 0.9 t

Generalised Open Maps

Generalised open maps situation:

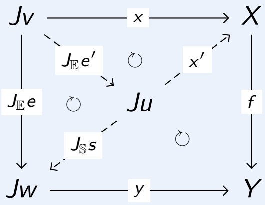
A generalised open maps situation on $\mathbb{M}$ is the following data:

- a set V together with a function $J\colon V \to \mathrm{ob}(\mathbb{M})$,
- two small categories $\mathbb{E}$ and $\mathbb{S}$ whose sets of objects are V,
- two functors $J_{\mathbb{E}}\colon \mathbb{E} \to \mathbb{M}$ and $J_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{M}$ coinciding with J on objects.

Ex: usual open maps are for $\mathbb{E} = \mathbb{P}$ and $\mathbb{S} = |\mathbb{P}|$.

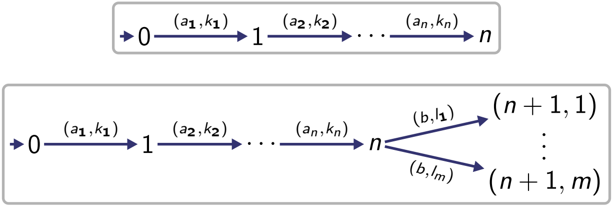
Generalised open maps

We say that a morphism $f : X \longrightarrow Y$ is $(\mathbb{E}, \mathbb{S})$-open if:

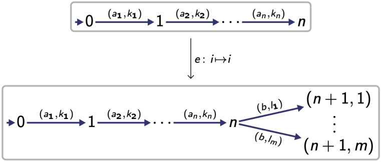


Generalised Open Maps Situation for Weighted Systems

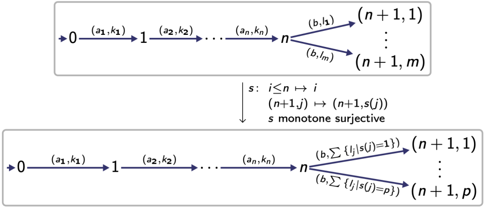
Two types of "paths":



Extensions ($\mathbb{E}_K$)



Mergings ($\mathbb{S}_K$)



Generalised Open $\simeq$ Coalgebraic, for Weighted Systems

Theorem

If K is a positive strict rearrangement monoid, then for a morphism of $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{W}_K(\_ \times A), *)$ whose domain is reachable, the following assertions are equivalent:

- it is $(\mathbb{E}_K, \mathbb{S}_K)$-open,
- it is a coalgebra homomorphism.

Ex: $K = \mathbb{R}_+, \mathbb{Q}_+, \mathbb{N}$, or any distributive lattice

Theorem

If K is a positive strict rearrangement monoid, then two pointed coalgebras of $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{W}_K(\_ \times A), *)$ are $(\mathbb{E}_K, \mathbb{S}_K)$-open bisimilar iff they are coalgebraically bisimilar.

Generalised Open $\simeq$ Coalgebraic, for Generative Probabilistic Systems

We can define a generalised open map situation $(\mathbb{E}_{gps}, \mathbb{S}_{gps})$ on $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{D}_{\leq 1}(\_ \times A), *)$ by restricting $(\mathbb{E}_{\mathbb{R}_+}, \mathbb{S}_{\mathbb{R}_+})$.

Theorem

For a morphism of $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{D}_{\leq 1}(\_ \times A), *)$ whose domain is reachable, the following assertions are equivalent:

- it is $(\mathbb{E}_{gps}, \mathbb{S}_{gps})$-open,
- it is a coalgebra homomorphism.

Theorem

Two pointed coalgebras of $\mathbf{Coal}_{\mathsf{lax}}(\mathcal{D}_{\leq 1}(\_ \times A), *)$ are $(\mathbb{E}_{gps}, \mathbb{S}_{gps})$-open bisimilar iff they are coalgebraically bisimilar.

Conclusion

- Presented here:
  - No open maps situations for generative probabilistic systems.
  - A generalisation of open maps to allow merging in addition to extensions.
  - A generalised open map situation for weighted and generative probabilistic systems.
- In the paper, but not presented:
  - Two notions of bisimulation relations (strong path and path) associated with generalised open maps.
  - A generalised open map situation such that:
    - ⋆ open bisim. $\simeq \exists$ strong path bisim. $\simeq \exists$ branching bisim.
    - ⋆ $\exists$ path bisim. $\simeq \exists$ weak bisim.
- Future work:
  - New Instances: Quantitative Petri Nets? Higher Dimensional Automata?
  - Which branching types are we capturing?