Aczel-Mendler Bisimulations in a Regular Category

CALCO'23, Indiana University Bloomington

Jérémy Dubut

National Institute of Advanced Industrial Science and Technology, Tokyo, Japan

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Let's Start Easy:

LTSs, Strong Bisimulations, and Composition

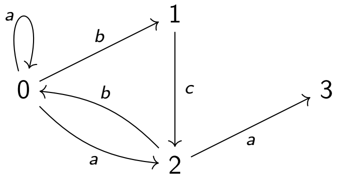
Transition Systems

Labelled Transition System:

A **TS** $T = (Q, \Delta)$ on the alphabet Σ is the following data:

- a set Q (of **states**) and
- a set of **transitions** $\Delta \subseteq Q \times \Sigma \times Q$.

- $\Sigma = \{a, b, c\}$,
- $Q = \{0, 1, 2, 3\}$,
- $\Delta = \{(0, a, 0), (0, b, 1), (0, a, 2), (1, c, 2), (2, b, 0), (2, a, 3)\}$.



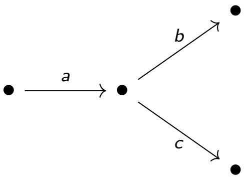
Strong Bisimulations of Transition Systems

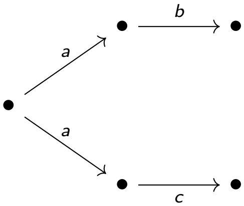
Strong Bisimulations **[Park81]**:

A **strong bisimulation** between $T_1 = (Q_1, \Delta_1)$ and $T_2 = (Q_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$ and

(ii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.

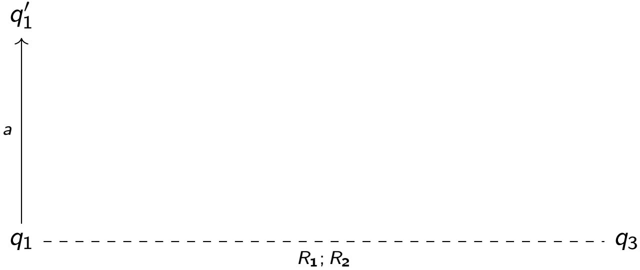




Strong Bisimulations are Closed under Composition

R_1 strong bisimulation between T_1, T_2 and R_2 strong bisimulation between T_2, T_3

$R_1; R_2 = \{(q_1, q_3) \mid \exists q_2. (q_1, q_2) \in R_1 \wedge (q_2, q_3) \in R_2\}$ strong bisimulation between T_1, T_3:



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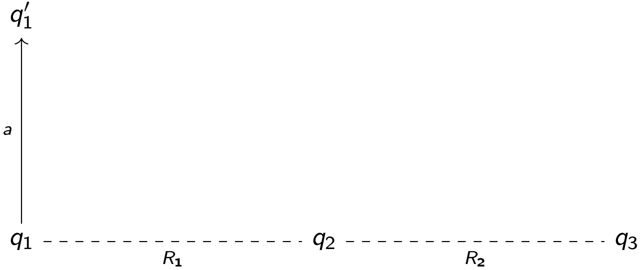

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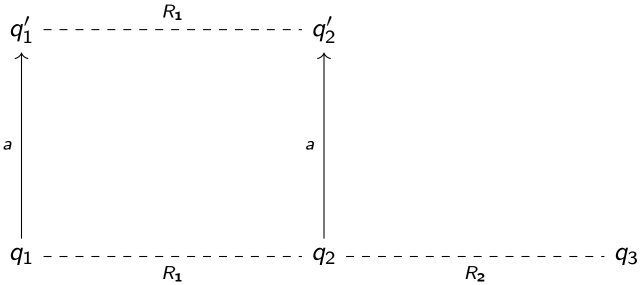

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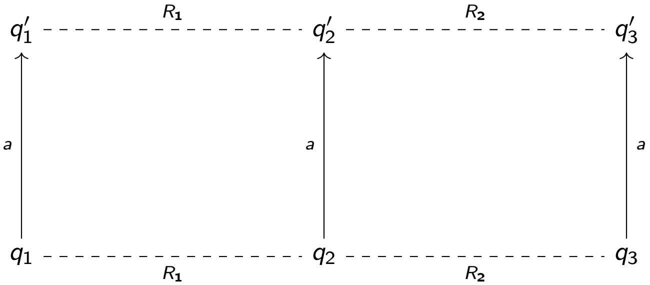

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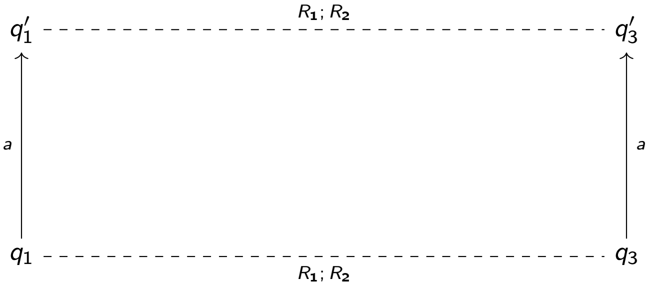


Aczel-Mendler Bisimulations of Coalgebras

Transition systems, as coalgebras

Set of transitions, as functions:

There is a bijection between sets of transitions $\Delta \subseteq Q \times \Sigma \times Q$ and functions of type:

$$\delta : Q \longrightarrow \mathcal{P}(\Sigma \times Q)$$

where $\mathcal{P}(X)$ is the powerset $\{U \mid U \subseteq X\}$.

Coalgebras:

Given an endofunctor $G : \mathcal{C} \longrightarrow \mathcal{C}$, a **coalgebra** is the following data:

- an object $Q \in \mathcal{C}$ and
- a morphism $\delta : Q \longrightarrow G(Q)$ of $\mathcal{C}$.

For LTS: $\mathcal{C} = \mathbf{Set}$, $G = X \mapsto \mathcal{P}(\Sigma \times X)$

Relations in a Category

Subobjects:

There is a preorder on monos with codomain X given by:

$$(u : U \rightarrowtail X) \sqsubseteq (v : V \rightarrowtail X) \quad \Leftrightarrow \quad \exists w : U \rightarrowtail V.\, u = v \cdot w.$$

A subobject of X is an equivalence class of monos $u : U \rightarrowtail X$ modulo $\sqsubseteq \cap \sqsupseteq$.

Ex: in **Set**, subobjects are subsets, and $\sqsubseteq$ is the inclusion

Relations:

A relation R from X to Y is a subobject of $X \times Y$.

Categories with Nice Relations: Regular Categories

Regular Categories:

A regular category is a finitely-complete category with a pullback-stable image factorization. In particular, it means it has a functorial pullback-stable (regular epi, mono)-factorization.

Ex: **Set**, any (quasi)topos, any abelian category, **Stone**, ...

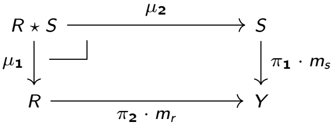
Allegories of Relations:

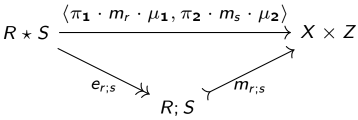
Given a regular category $\mathcal{C}$, then objects of $\mathcal{C}$ and relations between them form an allegory $\mathbf{Rel}(\mathcal{C})$, i.e.:

- it is a locally ordered 2-category,
- it has an anti-involution $(\_)^\dagger : \mathbf{Rel}(\mathcal{C})^{\mathsf{op}} \to \mathbf{Rel}(\mathcal{C})$,
- local posets are meet-semilattices,
- it satisfies the modular law $(R; S) \cap T \sqsubseteq (R \cap (T; S^\dagger)); S$.

Composition of Relations

Take two relations $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail Y \times Z$, the composition $m_{r;s} : R; S \rightarrowtail X \times Z$ is given by:





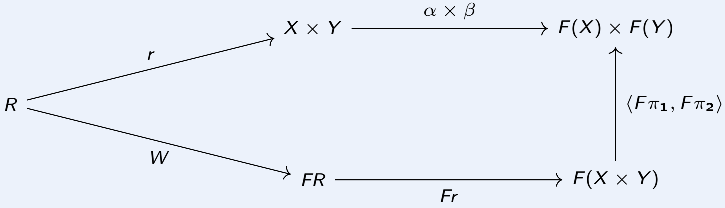
In **Set**:

- $R \star S = \{(x, y, z) \mid (x, y) \in R \wedge (y, z) \in S\}$,
- $R \star S \to X \times Z$ is given by $(x, y, z) \mapsto (x, z)$, and
- the image $R; S$ is $\{(x, z) \mid \exists y \in Y. (x, y) \in R \wedge (y, z) \in S\}$.

Aczel-Mendler Bisimulations

Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is an AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a morphism $W : R \longrightarrow FR$ (witness) such that:



In **Set**, for $F : X \mapsto \mathcal{P}(\Sigma \times X)$, AM-bisimulations are strong bisimulations:

- Fix $(x, y) \in R$, and $(a, x') \in \alpha(x)$.
- Commutativity means $(a, x') \in F(\pi_1 \cdot r) \cdot W(x, y)$, that is, there is y' such that $(a, (x', y')) \in W(x, y) \subseteq \Sigma \times R$, and $(x', y') \in R$.
- Commutativity means $(a, y') \in F(\pi_2 \cdot r) \cdot W(x, y) = \beta(y)$.

AM-Bisimulations are Closed under Composition?

Closure under composition:

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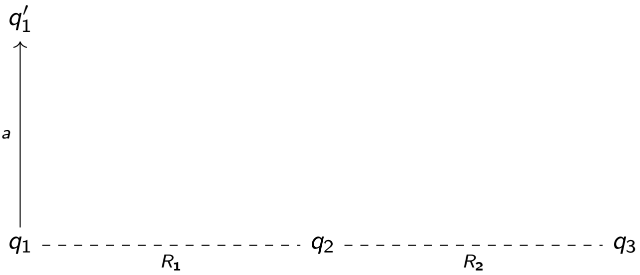


That was a choice of an intermediate state!

Proof

Starting with

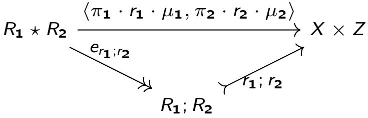
$$\begin{array}{ccccc} & \overset{r_1}{\nearrow} & X \times Y & \xrightarrow{\alpha \times \beta} & F(X) \times F(Y) \\ R_1 & & & & \uparrow \langle F\pi_1, F\pi_2 \rangle \\ & \underset{W_1}{\searrow} & FR_1 & \xrightarrow[Fr_1]{} & F(X \times Y) \end{array} \qquad \begin{array}{ccccc} & \overset{r_2}{\nearrow} & Y \times Z & \xrightarrow{\beta \times \gamma} & F(Y) \times F(Z) \\ R_2 & & & & \uparrow \langle F\pi_1, F\pi_2 \rangle \\ & \underset{W_2}{\searrow} & FR_2 & \xrightarrow[Fr_2]{} & F(Y \times Z) \end{array}$$

we want W such that

$$\begin{array}{ccccc} & \overset{r_1; r_2}{\nearrow} & X \times Z & \xrightarrow{\alpha \times \gamma} & F(X) \times F(Z) \\ R_1; R_2 & & & & \uparrow \langle F\pi_1, F\pi_2 \rangle \\ & \underset{W}{\searrow} & F(R_1; R_2) & \xrightarrow[F(r_1; r_2)]{} & F(X \times Z) \end{array}$$

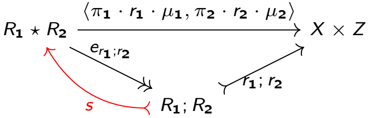
Proof

By definition of the composition



Proof

By definition of the composition and the regular axiom of choice



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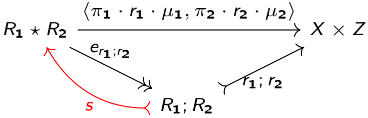

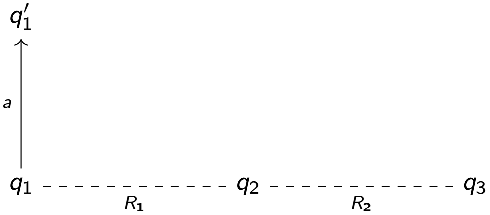

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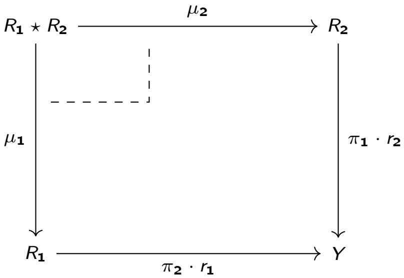
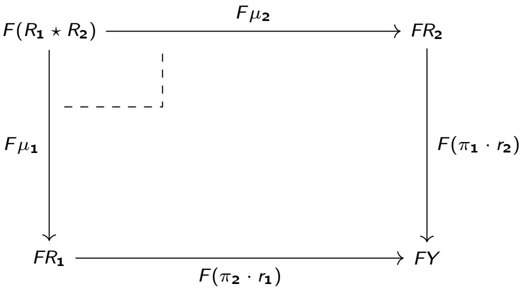
Proof

By definition of the composition, this is a (weak) pullback



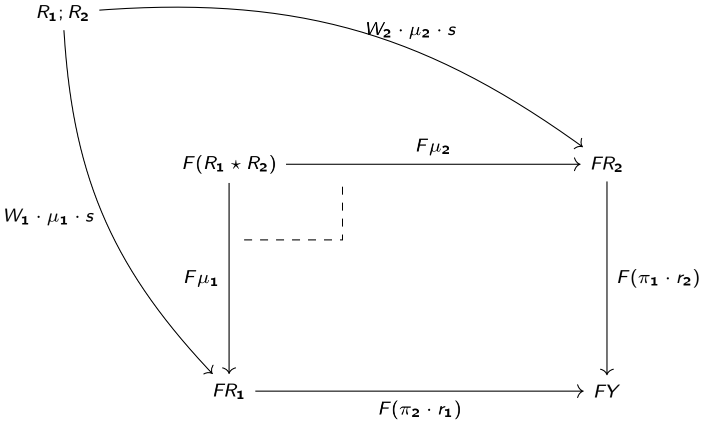
Proof

By preservation of weak pullback, this is a weak pullback



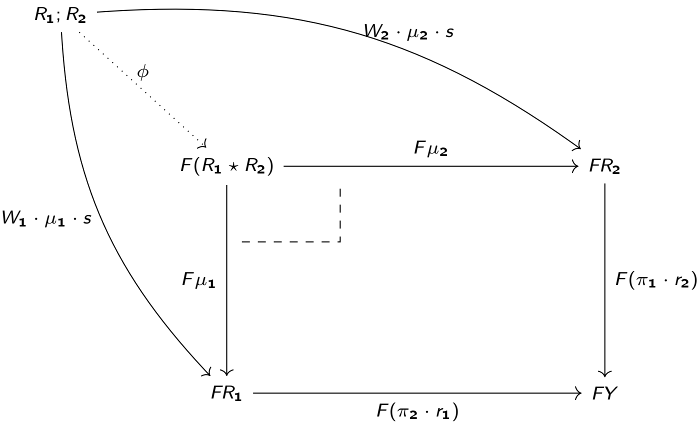
Proof

Putting everything together, the following commutes



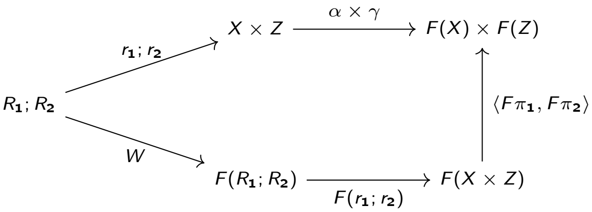
Proof

Then there is ϕ making the triangles commute



Proof

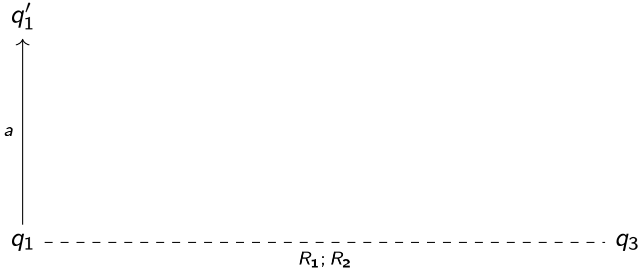
Choosing $W = F(e_{r1;r2}) \cdot \phi$ gives what we want:



From Picking to Collecting

Regular AM-Bisimulations

Picking vs Collecting



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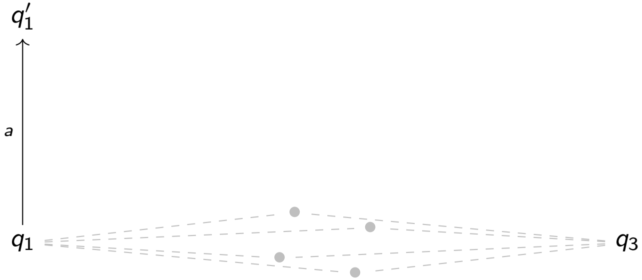

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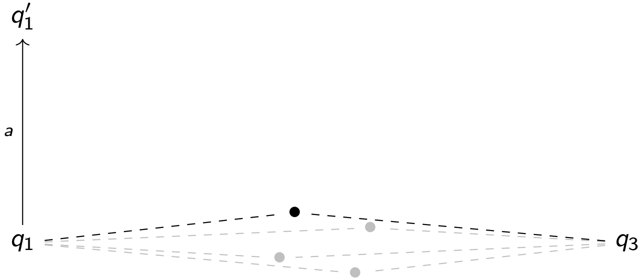
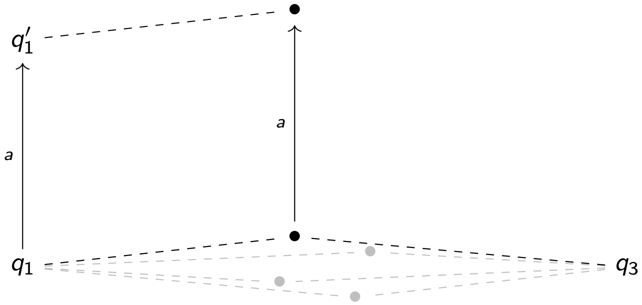


Pick one

Picking vs Collecting

Make the proof for this one



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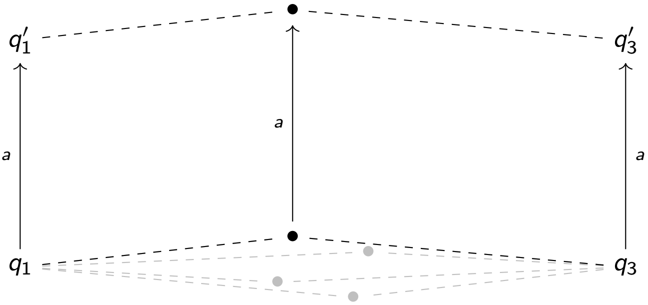

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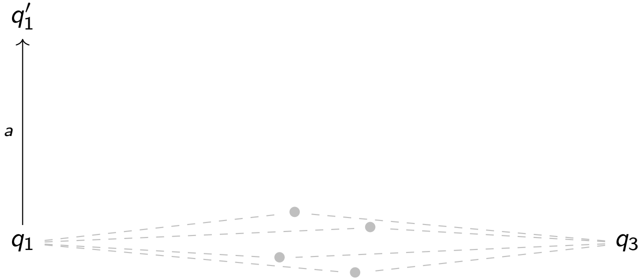

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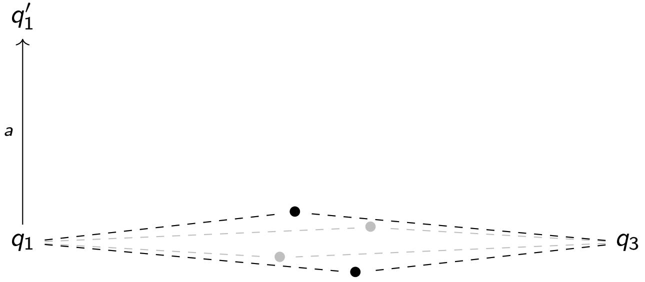
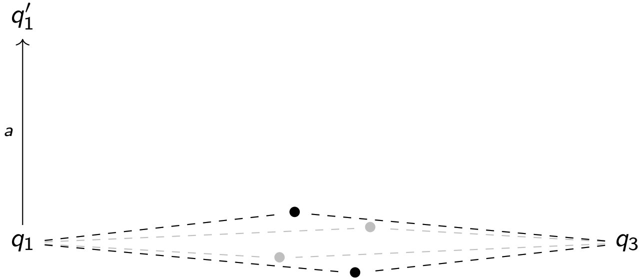


Collect many

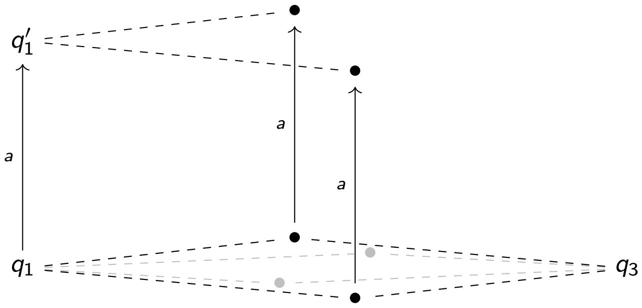
Picking vs Collecting



Collect many
(Make sure there is at least one)

Picking vs Collecting

Make the proof for all of them



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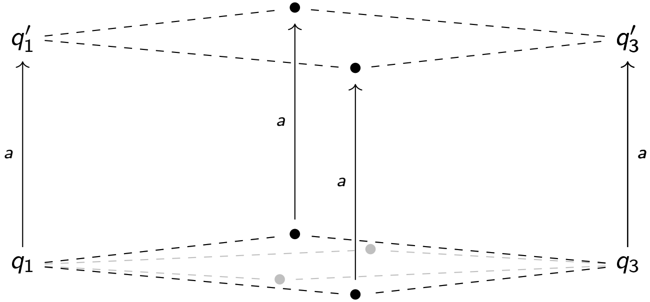

Collect many
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How to do that, abstractly?

Instead of building a *witness function*:

$$W : R \longrightarrow FR$$

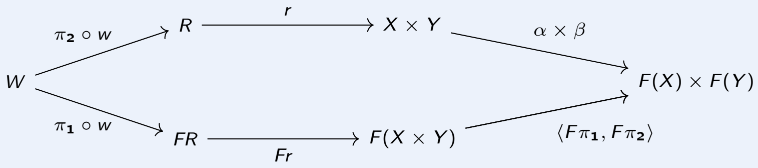
build a *witness relation*:

$$w \ : \ W \rightarrowtail R \times FR$$

Regular AM-Bisimulations

Regular Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is a regular AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a relation $w : W \rightarrowtail FR \times R$ (witness) such that $\pi_2 \cdot w$ is a regular epi and :



Basic Properties

Regular AM-Bisimulations Form a Dagger 2-Poset:

- Diagonals are regular AM-bisimulations.
- Regular AM-bisimulations are closed under inverse.
- When F covers pullbacks, regular AM-bisimulations are closed under composition.

Coincidence under the Axiom of Choice:

When $\mathcal{C}$ has the regular axiom of choice, then regular AM-bisimulations coincide with AM-bisimulations.

Relationship with Other Coalgebraic Bisimulations:

- Regular AM-bisimulations coincide with Hermida-Jacobs bisimulations.
- When F covers pullbacks, then behavioral equivalences are AM-bisimulations.
- When $\mathcal{C}$ has pushouts, every regular AM-bisimulation is included in a behavioral equivalence.

Example: Vietoris Bisimulations in **Stone**

Objects: Stone spaces, i.e., compact totally disconnected spaces

Morphisms: continuous functions

This is a regular category with pushouts

Subobjects: **closed** subsets

Vietoris functor $\mathcal{V}$: $X \mapsto$ set of closed subsets of X with a suitable topology

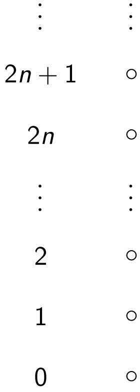
This endofunctor covers pullbacks (but do not preserve weak-pullbacks!)

[Bezhanishvili et al.'10]

Fix a Stone space A. Descriptive models coincide with $\mathcal{V}(\_) \times A$-coalgebras. Vietoris bisimulations coincide with HJ-bisimulations (so with regular AM too), but not with plain AM-bisimulations.

Counter-example from [Bezhanishvili et al.'10]

We want to construct two coalgebras $X \mapsto \mathcal{V}(X) \times \mathcal{P}(\mathbb{N} \times \{+, -\})$ in **Stone**



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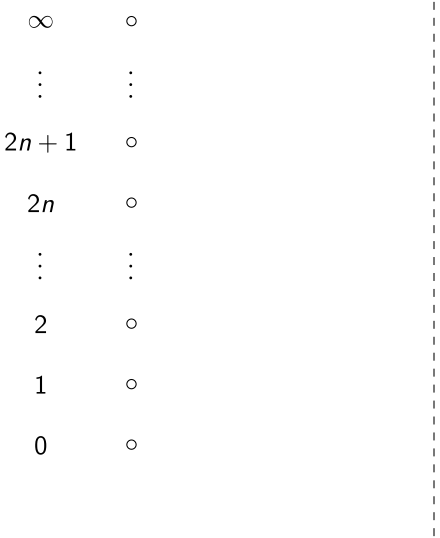

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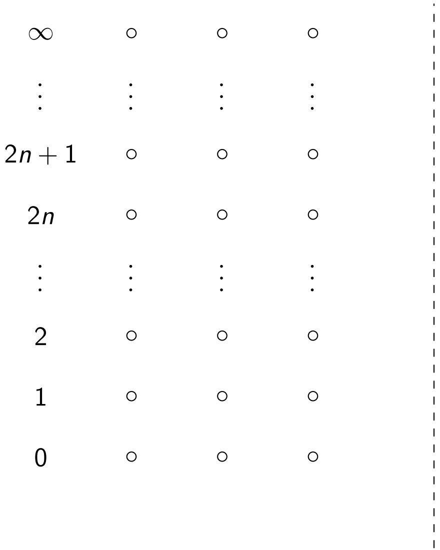

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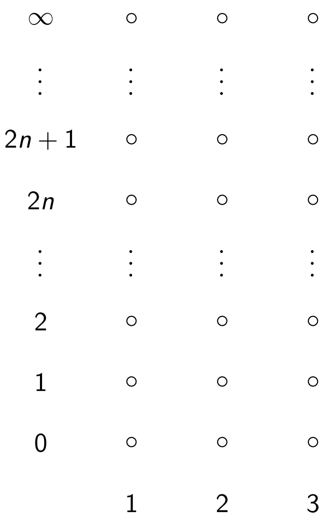

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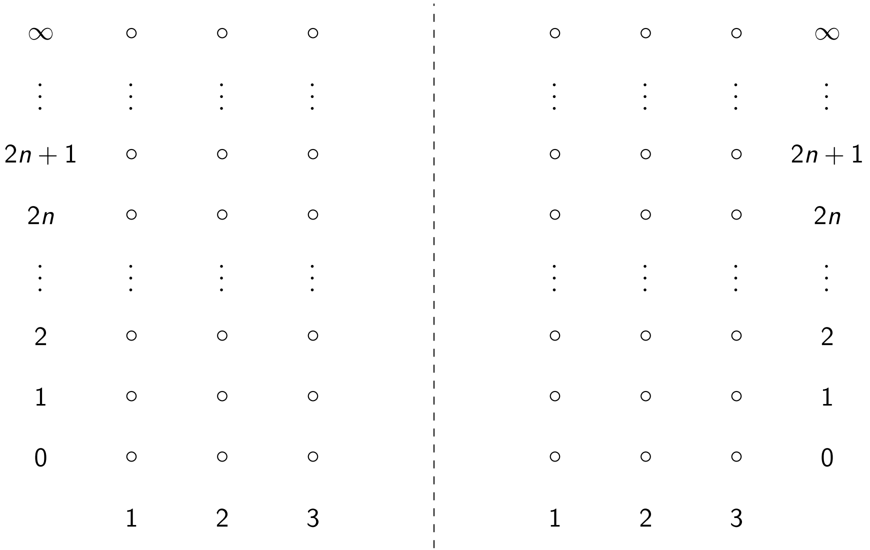

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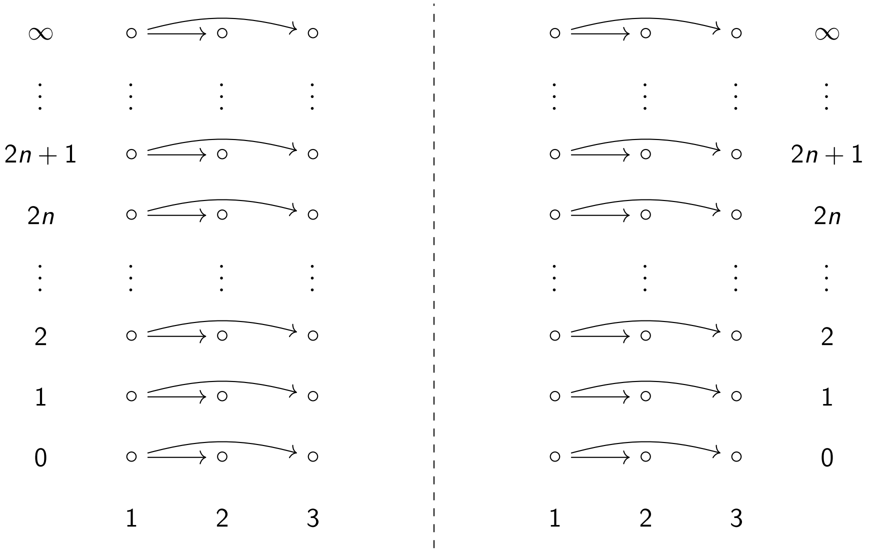

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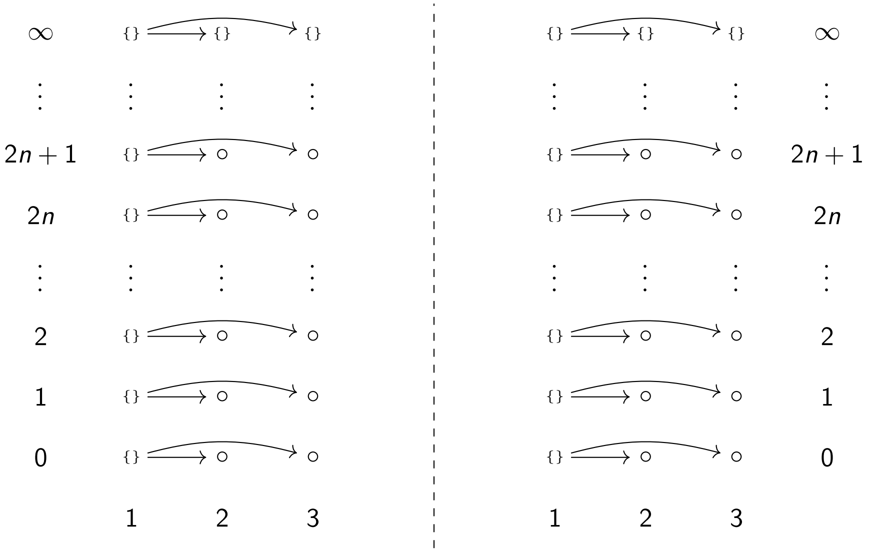

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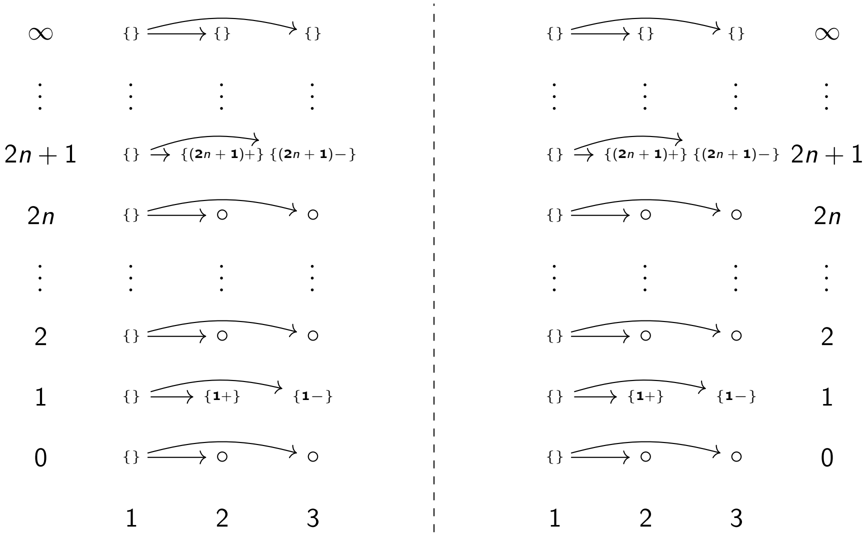

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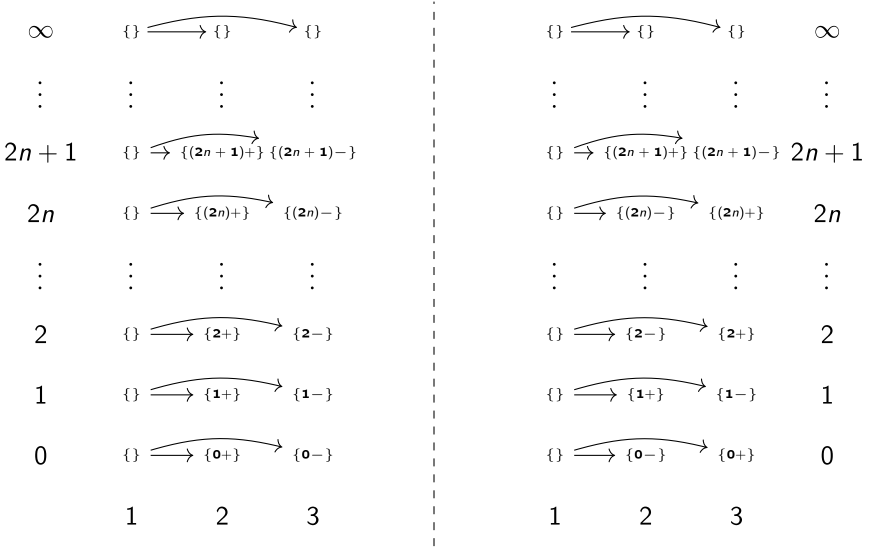

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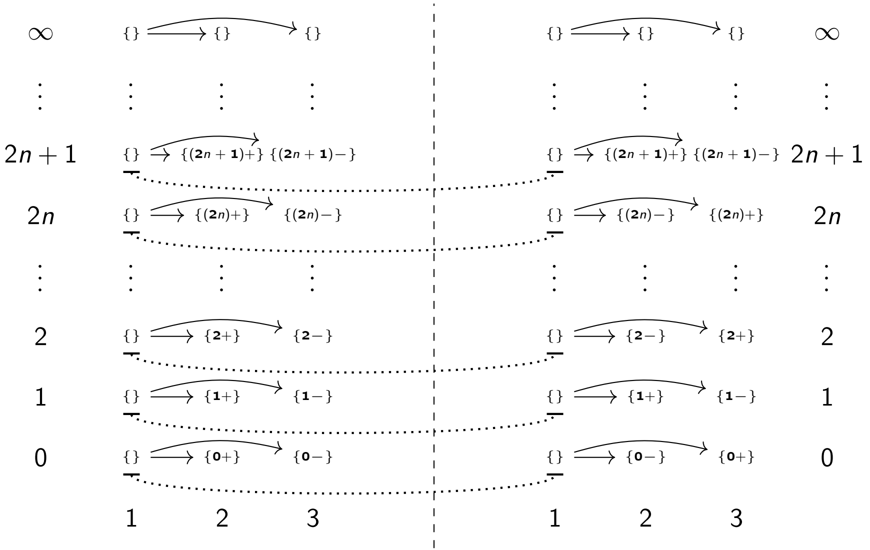

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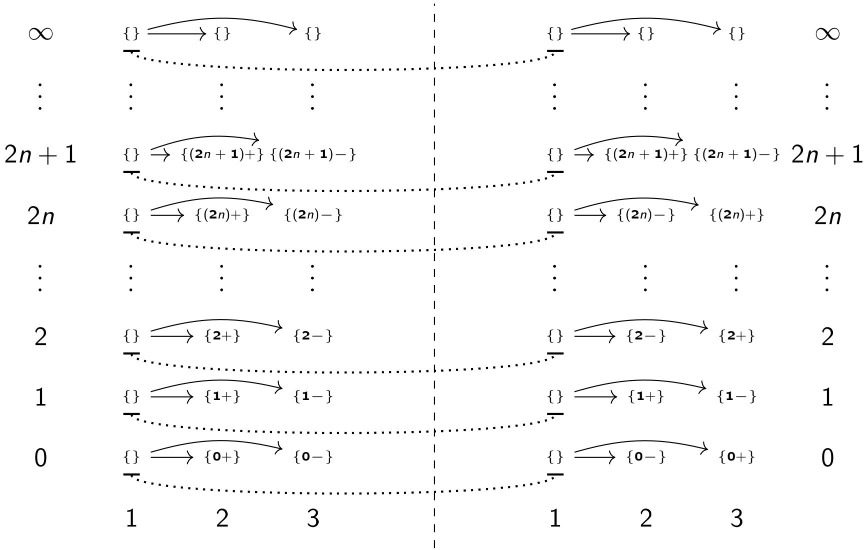

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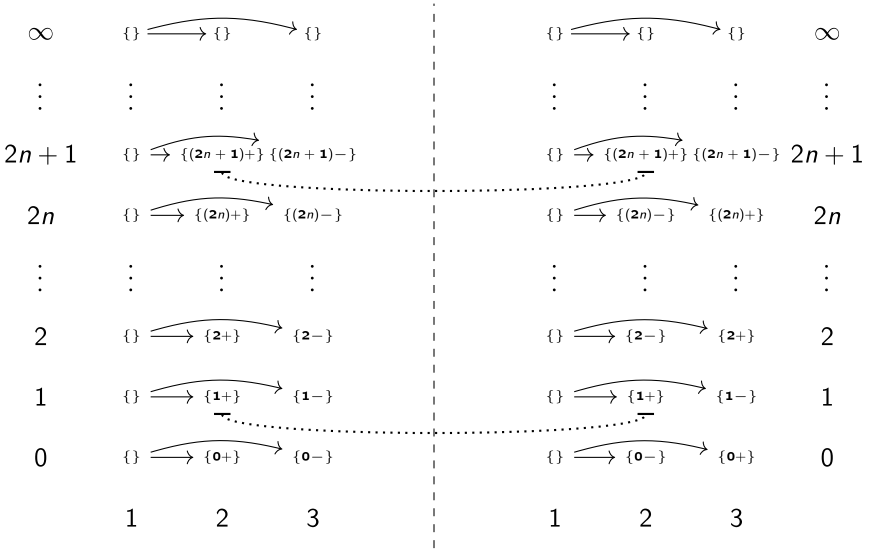

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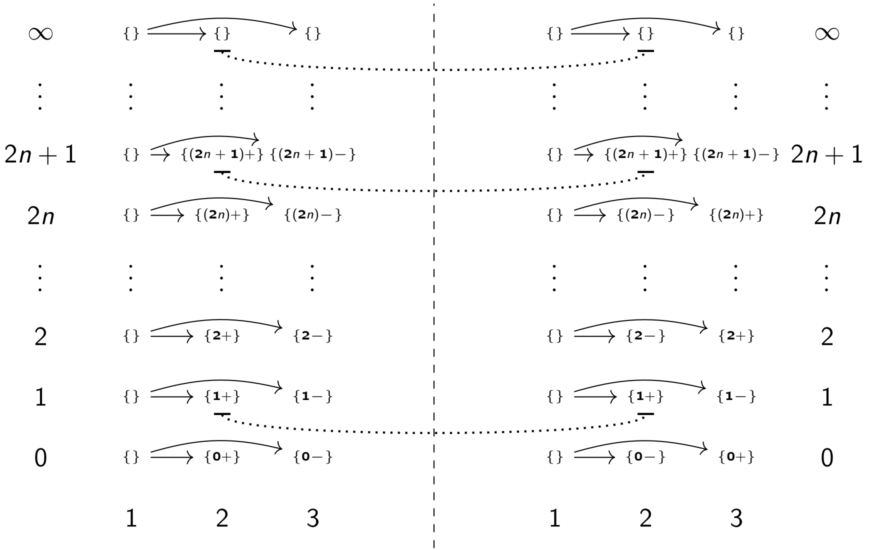

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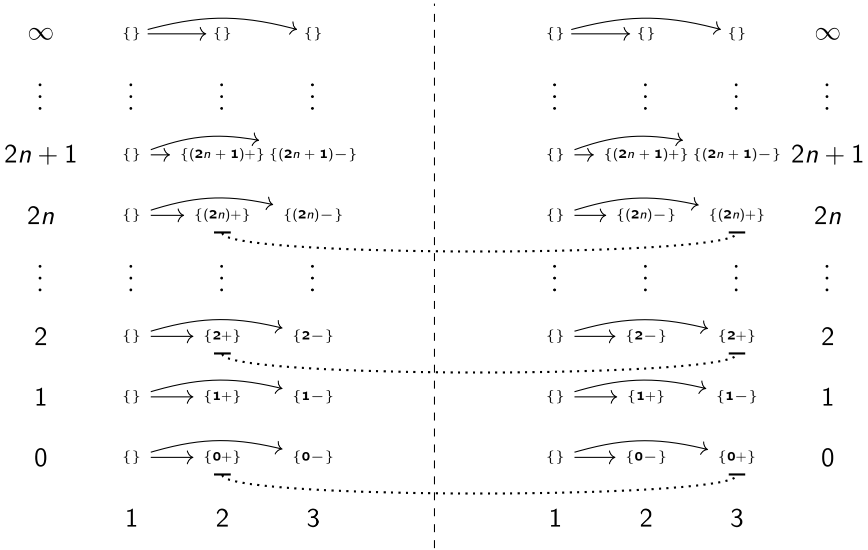

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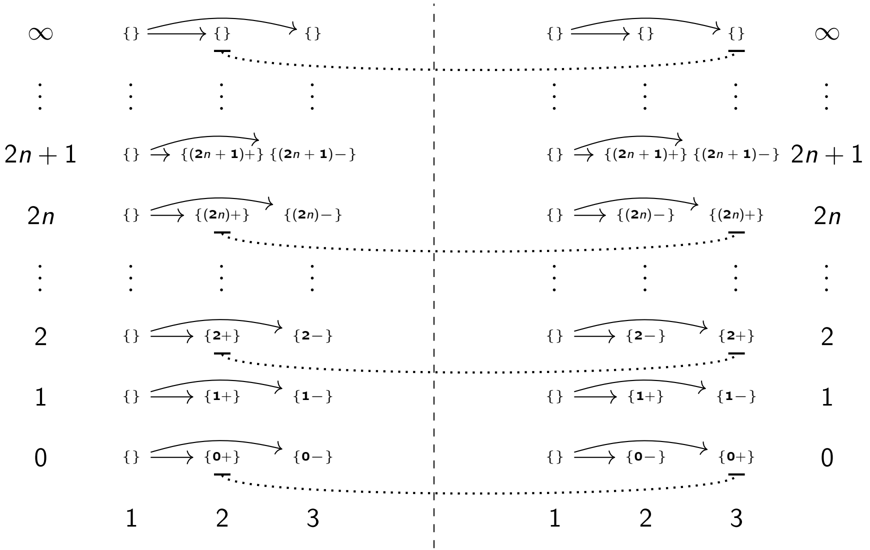

Counter-example from [Bezhanishvili et al.'10]

Regular AM $\neq$ AM

The following closed relation:

$$R = \{(i_2, i_2), (i_3, i_3) \mid i \in \mathbb{N} \text{ odd}\} \cup \{(i_2, i_3), (i_3, i_2) \mid i \in \mathbb{N} \text{ even}\}$$
$$\cup \{(i_1, i_1) \mid i \in \mathbb{N} \cup \{\infty\}\}$$
$$\cup \{(\infty_j, \infty_k) \mid j, k \in \{2, 3\}\}$$

is a regular AM-bisimulation but not an AM-bisimulation.

Proof:

The following is a witness closed relation $W \subseteq R \times (\mathcal{V}(R) \times \mathcal{P}(\mathbb{N} \times \{+, -\}))$:

$$W = \{((i_1, i_1), \{(i_2, i_2), (i_3, i_3)\}, \{\}) \mid i \in \mathbb{N} \text{ odd}\}$$
$$\cup \{((i_1, i_1), \{(i_2, i_3), (i_3, i_2)\}, \{\}) \mid i \in \mathbb{N} \text{ even}\}$$
$$\cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_2), (\infty_3, \infty_3)\}, \{\})\}$$
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(∞_1, ∞_1) has two witnesses {

The Special Case of Toposes

Toposes, as Relation Classifiers

Topos:

A topos is a finitely complete category $\mathcal{C}$ with **power objects**, that is, for every object X, there is a mono $\in_X : E_X \rightarrowtail X \times \mathcal{P}X$ such that for every relation $r : R \rightarrowtail X \times Y$ there is a unique morphism $\xi_r : Y \longrightarrow \mathcal{P}X$ such that there is a pullback of the form:

$$\begin{array}{ccc} R & \xrightarrow{\theta_r} & E_X \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle \in_X} \\ X \times Y & \xrightarrow[\mathrm{id} \times \xi_r]{} & X \times \mathcal{P}X \end{array}$$

In **Set**: $\mathcal{P}$ = power set, $E_X = \{(x, U) \mid x \in U\}$

The subobject classifier is $\Omega = \mathcal{P}\mathbf{1}$ and $\mathcal{P}X = \Omega^X$

This formulation implies cartesian closure

Folklore and More

Folklore:

$\mathcal{P}$ is a commutative monad whose Kleisli category is isomorphic to the category of relations of $\mathcal{C}$.

[Goy et al'21]

- For every endofunctor F of a topos $\mathcal{C}$ and object X of $\mathcal{C}$, there is a canonical morphism

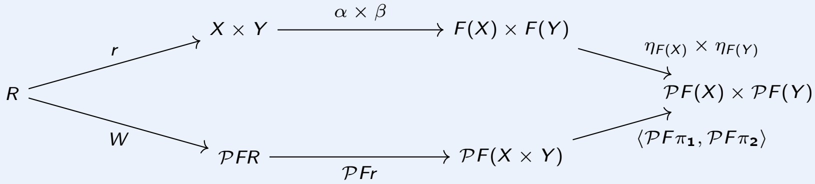
$$\sigma_{F,X} \colon \; F\mathcal{P}X \to \mathcal{P}FX.$$

- When F preserves weak pullbacks and epis, this is a natural transformation.
- If F is additonally a monad whose multiplication is weak cartesian, σ_F is a weak distributive law.
- If additionally the unit is also weak cartesian, then σ_F is a distributive law.
- In particular, for any non-trivial topos, $\sigma_{\mathcal{P}}$ is a weak distributive law but not a strict one.

A Nicer Formulation of Regular AM-Bisimulations

Toposal Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is a toposal AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a morphism $W : R \longrightarrow \mathcal{P}FR$ (witness) such that:



Basically, F-toposal-AM $= \mathcal{P}F$-AM

Toposal = Regular

In a topos, toposal AM-bisimulations coincide with regular AM-bisimulations.

Conclusion

- In this talk:
  - Plain AM-bisimulations work only with the axiom of choice.
  - Replacing witness functions by relations $\to$ regular AM-bisimulations
  - They work without axiom of choice:
    - closure under composition,
    - coincidence with HJ-bisimulations, behavioral equivalences.
  - They are reworded nicely in toposes.
- Not in this talk, but in the paper:
  - Allegory maps that are (toposal) AM-bisimulations are ($\mathcal{P}$)F-coalgebra homomorphisms.
  - Everything can be done for simulations too.
  - More examples (toposes for name-passing, weighted systems in categories of modules)
- Future work:
  - Relation with the $\neg\neg$-closure.
  - Regular AM-bisimulations for continuous probabilistic systems?
  - Regular AM-bisimulations in realizability toposes?