Complete non-orders and fixed points

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Introduction

• Interactive Theorem Proving is appreciated for reliability
• But it's also engineering tool for mathematics (esp. Isabelle/jEdit)
  • refactoring proofs and claims
  • sledgehammer
  • quickcheck/nitpick/nunchaku
• We develop an Isabelle library of order theory (as a case study)
  ⇒ we could generalize many known results, like:
  • completeness conditions: duality and relationships
  • Knaster-Tarski fixed-point theorem
  • Kleene's fixed-point theorem
Order

A binary relation (≡)

• **reflexive** ⇔ $x ≡ x$

• **transitive** ⇔ $x ≡ y$ and $y ≡ z$ implies $x ≡ z$

• **antisymmetric** ⇔ $x ≡ y$ and $y ≡ x$ implies $x = y$

• **partial order** ⇔ reflexive + transitive + antisymmetric
Order

A binary relation (\(\sqsubseteq\))

locale less_eq_syntax = fixes less_eq :: 'a \Rightarrow 'a \Rightarrow bool (infix "\(\sqsubseteq\)" 50)

• reflexive \(\iff\) \(x \sqsubseteq x\)
  locale reflexive = ... assumes "x \sqsubseteq x"

• transitive \(\iff\) \(x \sqsubseteq y\) and \(y \sqsubseteq z\) implies \(x \sqsubseteq z\)
  locale transitive = ... assumes "x \sqsubseteq y \Rightarrow y \sqsubseteq z \Rightarrow x \sqsubseteq z"

• antisymmetric \(\iff\) \(x \sqsubseteq y\) and \(y \sqsubseteq x\) implies \(x = y\)
  locale antisymmetric = ... assumes "x \sqsubseteq y \Rightarrow y \sqsubseteq x \Rightarrow x = y"

• partial order \(\iff\) reflexive + transitive + antisymmetric
  locale partial_order = reflexive + transitive + antisymmetric
Quasi-order

A binary relation (\(\sqsubseteq\))

```plaintext
locale less_eq_syntax = fixes less_eq :: 'a ⇒ 'a ⇒ bool (infix "\(\sqsubseteq\)" 50)
```

- **reflexive** \(\iff x \sqsubseteq x\)
  ```plaintext
  locale reflexive = ... assumes "x \sqsubseteq x"
  ```

- **transitive** \(\iff x \sqsubseteq y \text{ and } y \sqsubseteq z \text{ implies } x \sqsubseteq z\)
  ```plaintext
  locale transitive = ... assumes "x \sqsubseteq y ⇒ y \sqsubseteq z ⇒ x \sqsubseteq z"
  ```

- **antisymmetric** \(\iff x \sqsubseteq y \text{ and } y \sqsubseteq x \text{ implies } x = y\)
  ```plaintext
  locale antisymmetric = ... assumes "x \sqsubseteq y ⇒ y \sqsubseteq x ⇒ x = y"
  ```

- **quasi-order** \(\iff\) reflexive + transitive
  ```plaintext
  locale quasi_order = reflexive + transitive
  ```
Pseudo-order [Skala 1971]

A binary relation (⊆)

   locale less_eq_syntax = fixes less_eq :: 'a ⇒ 'a ⇒ bool (infix "⊆" 50)

• reflexive ⇔ x ⊆ x
   locale reflexive = ... assumes "x ⊆ x"

• transitive ⇔ x ⊆ y and y ⊆ z implies x ⊆ z
   locale transitive = ... assumes "x ⊆ y ⇒ y ⊆ z ⇒ x ⊆ z"

• antisymmetric ⇔ x ⊆ y and y ⊆ x implies x = y
   locale antisymmetric = ... assumes "x ⊆ y ⇒ y ⊆ x ⇒ x = y"

• pseudo order ⇔ reflexive + antisymmetric
   locale pseudo_order = reflexive + antisymmetric
Non-order

A binary relation (\(\sqsubseteq\))

```plaintext
locale less_eq_syntax = fixes less_eq :: 'a ⇒ 'a ⇒ bool (infix "\(\sqsubseteq\)" 50)

• reflexive \(\iff\) \(x \sqsubseteq x\)
  locale reflexive = ... assumes "x \sqsubseteq x"

• transitive \(\iff\) \(x \sqsubseteq y\) and \(y \sqsubseteq z\) implies \(x \sqsubseteq z\)
  locale transitive = ... assumes "x \sqsubseteq y ⟹ y \sqsubseteq z ⟹ x \sqsubseteq z"

• antisymmetric \(\iff\) \(x \sqsubseteq y\) and \(y \sqsubseteq x\) implies \(x = y\)
  locale antisymmetric = ... assumes "x \sqsubseteq y ⟹ y \sqsubseteq x ⟹ x = y"
```
Locale combinations
Complete non-orders

- **upper/lower bounds:**
  - **definition** "bound (\(\sqsubseteq\)) \(X\) \(b\) \(\equiv\) \(\forall x \in X. x \sqsubseteq b\)"

- **greatest/least elements:**
  - **definition** "extreme (\(\sqsubseteq\)) \(X\) \(e\) \(\equiv\) \(e \in X \land (\forall x \in X. x \sqsubseteq e)\)"

- **suprema/infima** (l.u.b./g.l.b.):
  - **abbreviation** "extreme_bound (\(\sqsubseteq\)) \(X\) \(s\) \(\equiv\) extreme (\(\sqsupseteq\)) \(\{b. \text{bound (\(\sqsubseteq\))} \(X\) \(b\}\) \(s\)"

- **complete** \(\iff\) any set \(X\) of elements has a supremum
  - **locale** complete = **assumes** "\(\exists s. \text{extreme_bound (\(\sqsubseteq\))} \(X\) \(s\)"

**Proposition:** The dual of complete **non-order** is complete

**sublocale** complete \(\subseteq\) dual: complete "(\(\sqsupseteq\))"
Knaster–Tarski fixed points
Knaster–Tarski: Part 1

• **Theorem** (Knaster–Tarski, part 1)
  Any monotone map $f$ on a complete order $\sqsubseteq$ has a fixed point
  (monotone: $x \sqsubseteq y \implies f(x) \sqsubseteq f(y)$)
  (fixed point: $f(x) = x$)

• **Theorem** [Stauti & Maaden 2013]
  Any monotone map $f$ on a complete pseudo-order $\sqsubseteq$ has a fixed point
  (relaxed transitivity)

**Theorem** [this work]
Any monotone map $f$ on a complete non-order $\sqsubseteq$ has a quasi-fixed point
(relaxed transitivity, reflexivity, antisymmetry)

(quasi-fixed point: $f(x) \sim x$ i.e., $f(x) \sqsubseteq x$ and $x \sqsubseteq f(x)$)
Proof sketch (by Stauti & Maaden)

**definition** AA where "AA ≡
{A. f ` A ⊆ A ∧ (∀B ⊆ A. ∪ B ∈ A)}"

**lemma** "∃c ∈ ∩ AA. f c = c"

**proof**

**define** c where "c ≡ ∪ (∩ AA)"

**show** "c ∈ ∩ AA"...

**show** "f c = c"

**proof** (rule antisym)

**show** "f c ⊆ c"...

**show** "c ⊆ f c"...

qed

qed
Proof sketch (minus reflexivity)

**definition** AA where "AA ≡
{A. f ` A ⊆ A ∧ (∀B ⊆ A. ∪ B ∈ A)}"

**lemma** "∃c ∈ ∩ AA. f c = c"

**proof**

  **define c where** "c ≡ ∪ (∩ AA)"
  **show** "c ∈ ∩ AA"...
  **show** "f c = c"
  **proof** (rule antisym)
    **show** "f c ⊆ c"...
    **show** "c ⊆ f c"...

qeda
qed
Proof sketch (minus antisymmetry)

**Definition** \( AA \) where \( AA \equiv \{ A. f \notin A \subseteq A \land (\forall B \subseteq A. \forall s. \text{extreme_bound}(\subseteq) B s \rightarrow s \in A)\} \)

**Lemma** \( \exists c \in \bigcap AA. f c \sim c \)

**Proof**
- obtain \( c \) where \( \text{extreme_bound}(\subseteq) (\bigcap AA) c \)...
- show \( c \in \bigcap AA \)...
- show \( f c \sim c \)
- **proof (rule antisym)**
  - show \( f c \subseteq c \)...
  - show \( c \subseteq f c \)...
- qed
- qed

Supremum is not unique:

- \( f c \subseteq c \) and \( c \subseteq f c \) doesn't mean \( f c = c \)
Knaster–Tarski, Part 1: Existence

• Main result 1
  theorem (in complete)
  assumes "monotone (⊆) (⊆) f" shows "∃x. f x ~ x"
Knaster–Tarski, Part 2: Completeness

• **Theorem** (Knaster–Tarski, Part 2)
  For any monotone map on a complete partial order, the set of fixed points is complete

• **Theorem** [Stauti & Maaden 2013]
  Any monotone map on a complete pseudo order has a least fixed point

• **Conjecture**?
  Any monotone map on a complete non-order has a least quasi-fixed point?
Least quasi-fixed points?

• **Counterexample** [Nitpick]
  
  **Nontheorem (in complete)**
  
  Assumes "monotone (\(\sqsubseteq\)) (\(\sqsubseteq\)) \(f\)" shows "\(\exists p.\) extreme (\(\sqsupseteq\)) \{s. \(f \sim s\) \(\sim\) \} \(p\)"

**Nitpick**

\[
f = (\lambda x. \_)(a_1 := a_3, a_2 := a_3, a_3 := a_3, a_4 := a_1)
\]

(\(\sqsubseteq\)) = (\(\lambda x. \_\))

(\(a_1 := (\lambda x. \_)(a_1 := \text{False}, a_2 := \text{True}, a_3 := \text{True}, a_4 := \text{True})\),
\(a_2 := (\lambda x. \_)(a_1 := \text{True}, a_2 := \text{True}, a_3 := \text{True}, a_4 := \text{True})\),
\(a_3 := (\lambda x. \_)(a_1 := \text{True}, a_2 := \text{False}, a_3 := \text{True}, a_4 := \text{False})\),
\(a_4 := (\lambda x. \_)(a_1 := \text{True}, a_2 := \text{True}, a_3 := \text{True}, a_4 := \text{False})\))
least quasi-fixed points?

- **Counterexample** [Nitpick]

\[ f \subseteq a_1 \quad a_2 \quad a_3 \quad a_4 \]

Quasi-fixed points: \[ T = a_3 \]

Not least, as \[ a_3 \not\subseteq a_4 \]

\[ a_1 \not\subseteq a_1 \]

\[ a_4 \not\subseteq a_4 \]

\[ = \bot \]
Argument by Stauti & Maaden

**Definition** $AA$ where $AA \equiv \{ A. f \ ` A \subseteq A \land (\forall B \subseteq A. \bigcup B \in A) \}$

**Lemma** $\exists c \in \bigcap AA. f c = c$ ...

**Definition** $A$ where $A \equiv \{ a. \text{bound (\equiv)} \{ p. f p = p \} \ a \}$

**Lemma** $A \in AA$

**Proof**

1. Show $f ` A \subseteq A$...
2. Show $\forall B \subseteq A. \bigcup B \in A$...

qed

$c \in AA$!

So $c \in A \cap \text{FP}$

$FP = \{ p. f p = p \}$

i.e., least fixed point

$A = \text{(lower bounds of FP)}$

$\in AA$!
Proof of "f ` A ⊆ A"

QFP = \{p. f p \sim p\}

A = (lower bounds of QFP)

a is a lower bound

p is in QFP

if x \sim y \subseteq z \Rightarrow x \subseteq z

by monotonicity
Attractivity

locale semiattractive = assumes "x ⊆ y ⇒ y ⊆ x ⇒ y ⊆ z ⇒ x ⊆ z"
Attractivity

locale attractive =
semiattractive + dual: semiattractive "(⊇)"

sublocale transitive ⊆ attractive

sublocale antisymmetric ⊆ attractive
Knaster-Tarski, part 2: Completeness

- **Main result 2:**

  *Theorem* (in complete_attractive)

  *Assumes* "monotone ($\subseteq \subseteq$) f" *shows* "complete_in ($\subseteq$) \( \{p. f p \sim p\} \)"

\[ U = (\text{upper bounds of } A) \]

\[ \text{least qfp in } U \]

\[ q \sim f q \]

\[ A \subseteq \{p. f p \sim p\} \]
Knaster-Tarski, part 2: Completeness

**Main result 2:**

- **Theorem (in complete_attractive)**
  
  assumes "monotone (\(\sqsubseteq\) (\(\sqsubseteq\)) f" shows "complete_in (\(\sqsubseteq\)) \{p. f p \sim p\}"

- **In pseudo order, \(x \sim y \iff x = y\). So**
  
  **Corollary (in complete_pseudo_order)**
  
  assumes "monotone (\(\sqsubseteq\) (\(\sqsubseteq\)) f" shows "complete_in (\(\sqsubseteq\)) \{p. f p = p\}"

Completes Stauti & Maaden's work!

... but is reflexivity necessary?
Conjecture (in complete_antisymmetric) assumes "monotone (⊆) (⊆) f" shows "complete_in (⊆) {p. f p = p}"

U = (upper bounds of A)

least quasi-fixed point in U, but...

A ⊆ {p. f p = p}

there might be a smaller non-quasi fixed point!
Completeness only with antisymmetry

• a key lemma

lemma qfp_interpolant:
assumes "complete (⊆)"
and "monotone (⊆) (⊆) f"
and "∀a ∈ A. ∀b ∈ B. a ⊆ b"
and "∀a ∈ A. f a = a"
and "∀b ∈ B. f b = b"
shows "∃s. f s ≈ s ∧ (∀a ∈ A. a ⊆ s) ∧ (∀b ∈ B. s ⊆ b)"

• Main result 3

theorem (in complete_antisymmetric)
assumes "monotone (⊆) (⊆) f" shows "complete_in (⊆) {p. f p = p}"
Kleene fixed points
Kleene fixed points, part 1: Construction

• **Theorem** (Kleene, part 1)
  Let \( f \) be a Scott-continuous map on a directed-complete order. Then \( \bigsqcup_n f^n(\bot) \) exists and is a fixed point.

• **Theorem** [Mashburn 1983]
  Let \( f \) be an \( \omega \)-continuous map on a \( \omega \)-complete order. Then \( \bigsqcup_n f^n(\bot) \) exists and is a fixed point.

**Theorem** [this work]
Let \( f \) be an \( \omega \)-continuous map on a \( \omega \)-complete non-order. Let \( \bot \) be a least element. Then \( \{f^n(\bot) \mid n \in \mathbb{N}\} \) has suprema, and they are all quasi-fixed point.
ω-completeness

• **ω-chain**: a sequence $c_0, c_1, ...$ s.t. $i \leq j$ implies $c_i \sqsubseteq c_j$
  
  _definition_ "omega_chain C ≡ \( \exists c :: \text{nat} \Rightarrow \text{'a. monotone} (\leq) (\sqsubseteq) c \land C = \text{range } c)"

• **ω-complete**: any ω-chain has a supremum
  
  _locale_ omega_complete =
  
  _assumes_ "omega_chain C ⇒ \( \exists s. \text{extreme_bound} (\sqsubseteq) C s)"

• **ω-continuity**: $f$ preserves all suprema of ω-chains
  
  • _definition_ "omega_continuous f ≡ \( \forall C. \text{omega_chain } C \rightarrow \forall s. \text{extreme_bound} (\sqsubseteq) C s \rightarrow \text{extreme_bound} (\sqsubseteq) (f \circ C) (f s)"

ω-continuity implies "near" monotonicity

• lemma
  assumes "omega_continuous f" and "x ⊑ y" and "x ⊑ x" and "y ⊑ y"
  shows "f x ⊑ f y"

proof-
  have "omega_chain {x, y}"...
  have "extreme_bound {x, y} y"...
  have "extreme_bound (f ` {x, y}) (f y) using omega_continuity"...
  then show "f x ⊑ f y" by auto

qed
\( \{f^n(\perp) \mid n \in \mathbb{N}\} \) is an \( \omega \)-chain

By near monotonicity, \( f^2 \perp \subseteq f^2 \perp \) gives \( f \perp \subseteq f \perp \)

Because \( \perp \) is least, \( f \perp \subseteq f \perp \)

Because \( \perp \) is least, \( f^2 \perp \subseteq f^2 \perp \)

\( \cdots \) \( \omega \)-chain!
$\bigcup_n f^n(\bot)$ is quasi-fixed; as usual

by $\omega$-completeness

$\bigcup_n f^n(\bot)$

$f(\bigcup_n f^n(\bot))$
\( \bigcup_n f^n(\bot) \) is quasi-fixed; as usual

\( \bigcup_n f^n(\bot) \quad \bigcup_n f(f^n(\bot)) \)

\( \bot \quad f \bot \quad f^2 \bot \quad f^3 \bot \quad f^4 \bot \quad \ldots \)

\( \omega \)-continuity
Kleene fixed point, part 1: Construction

• Main result 4: 
  \textbf{theorem} shows \( \exists p. \text{extreme_bound} (\sqsubseteq) \{f^n(\bot) \mid n \in \mathbb{N}\} \ p \)
  \textbf{and} "extreme_bound (\sqsubseteq) \{f^n(\bot) \mid n \in \mathbb{N}\} \ p \implies f \ p \sim p"

  there is a supremum for \( \{f^n(\bot) \mid n \in \mathbb{N}\} \)

  and any such is a quasi-fixed point
Kleene fixed point, part 2: Leastness

- **Theorem** (Kleene, part 2)
  Let $f$ be a Scott-continuous map on a directed-complete order. Then $\bigcup_n f^n(\bot)$ is the **least** fixed point.

- **Theorem** [Mashburn 1983]
  Let $f$ be an $\omega$-continuous map on a $\omega$-complete order. Then $\bigcup_n f^n(\bot)$ is the **least** fixed point.

- **Conjecture**
  Let $f$ be an $\omega$-continuous map on a $\omega$-complete **non-order**. Are suprema of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ **least quasi**-fixed points?
Is $\bigcup_n f^n(\perp)$ least?

- **Counterexample** [Nitpick]

Attractivity is the key, again!
∪ₙ fⁿ(⊥) is least under attractivity

**fix** q **assume** "q ∼ f q" **have** "fⁿ ⊥ ⊆ q"

Due to attractivity:
q ∼ f q ⊆ q ⇒ q ⊆ q

attractivity  near monotonicity
\[ \bigcup_n f^n(\bot) \text{ is least under attractivity} \]

**fix q assume "q ⪰ f q" have "f^n \bot \sqsubseteq q" by ...**

**then show "\( \bigcup_n f^n(\bot) \sqsubseteq q \)"...**
Kleene fixed point, part 2

Main result 5 (last):

**corollary** (in attractive)
"extreme_bound (⊆) \{f^n(⊥) \mid n \in \mathbb{N}\} s \leftrightarrow \text{extreme (⊇) } \{\text{q. } f \sim q\} s"

suprema of \{f^n(⊥) \mid n \in \mathbb{N}\} are **the** least quasi-fixed points
Conclusion

• An Isabelle/HOL library for non-orders
• Generalized some (folklore) results on completeness
• Generalized Knaster—Tarski fixed-point theorem
  • monotone map on complete non-order has a quasi-fixed point
  • if attractive, the set of quasi-fixed points is complete
• Generalized Kleene fixed-point theorem
  • for an \( \omega \)-continuous map on \( \omega \)-complete non-order, suprema of \( \{f^n \perp | n \in \mathbb{N}\} \) is a quasi-fixed point
  • if attractive, they are the least quasi-fixed points