Natural homology
Computability and Eilenberg-Steenrod axioms

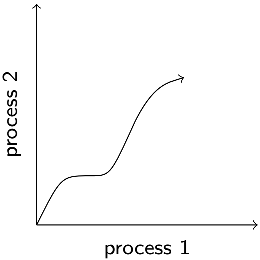
Applied and Computational Algebraic Topology
HIM, Bonn

Jérémy DUBUT- LSV, ENS Paris-Saclay

joint work with
Eric GOUBAULT - LIX, Ecole Polytechnique
Jean GOUBAULT-LARRECQ - LSV, ENS Paris-Saclay

5th May, 2017

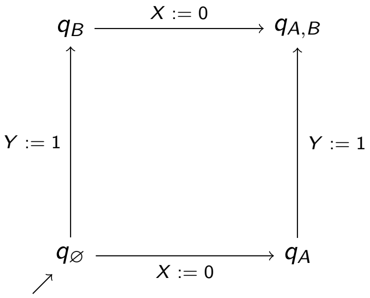
True concurrency



- Petri nets **[Petri 62]**
- progress graphs **[Dijkstra 68]**
- trace theories **[Mazurkiewicz 70s]**
- event structures **[Winskel 80s]**
- higher dimensional automata (HDA) **[Pratt 91]**

Interleaving vs continuity

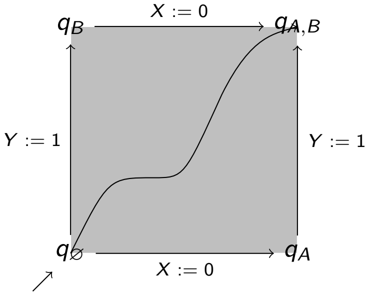
$$X := 0 \parallel Y := 1$$



Interleaving behaviors : A then B or B then A

Interleaving vs continuity

$$X := 0 \parallel Y := 1$$



Continuous behaviors : any scheduling of A and B

True concurrency, geometrically

truly concurrent system	topological space
states	points
executions	paths
modulo scheduling of independent actions	modulo homotopy

True concurrency, geometrically

truly concurrent system	topological space
states	points
executions	paths
modulo scheduling of independent actions	modulo homotopy

Problem : executions are directed, paths are not

True concurrency, geometrically

truly concurrent system	directed space
states	points
executions	directed paths
modulo scheduling of independent actions	modulo directed homotopy

D-spaces and dipaths **[Grandis 01]**

A **d-space** is a topological X with a subset $\overrightarrow{P}(X)$ of paths, called **dipaths**, such that :

- constant paths are dipaths,
- dipaths are closed under concatenation,

$$\begin{aligned} \gamma_1 \star \gamma_2(t) &= \gamma_1(2t) & \text{if } t \leq \frac{1}{2} \\ &= \gamma_2(2t-1) & \text{if } t \geq \frac{1}{2} \end{aligned}$$

- dipaths are closed under non-decreasing reparametrization, $\gamma \circ r$ with $r : [0,1] \longrightarrow [0,1]$ continuous monotonic.

The set of paths can be equipped with the compact-open topology $\overrightarrow{P}(X)$ and $\overrightarrow{P}(X)(a,b)$ can be equipped with the subspace topology

A **dimap** is a continuous function $f : X \longrightarrow Y$ such that for every $\gamma \in \overrightarrow{P}(X)$, $f \circ \gamma \in \overrightarrow{P}(Y)$.

The different d-space structures of the segment

$\overrightarrow{[0,1]}$: dipaths are monotonic paths,

$\overline{[0,1]}$: dipaths are constant paths,

$\overleftrightarrow{[0,1]}$: dipaths are all the paths.

Ex : dipaths of X = dimaps from $\overrightarrow{[0,1]}$ to X

Homotopy

A **homotopy** from γ to τ, paths from a to b, is a continuous function

$$H : [0,1] \times [0,1] \longrightarrow X$$

such that :

- $H(0, \_) = a$ and $H(1, \_) = b$,
- $H(\_, 0) = \gamma$ and $H(\_, 1) = \tau$.

Equivalently, it is a path in the space of paths $P(X)(a,b)$!

Two paths are **homotopic** if there is a homotopy between them, or equivalently, if they are in the same path-connected components of $P(X)(a,b)$.

Dihomotopy

A **dihomotopy** from γ to τ, dipaths from a to b, is a dimap

$$H : \overline{[0,1]} \times \overrightarrow{[0,1]} \longrightarrow X$$

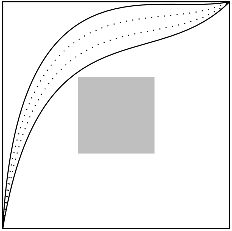
such that :

- $H(0, \_) = a$ and $H(1, \_) = b$,
- $H(\_, 0) = \gamma$ and $H(\_, 1) = \tau$.

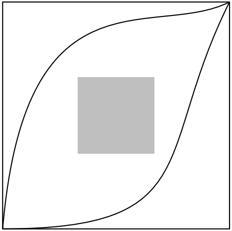
Equivalently, it is a path in the space of dipaths $\overrightarrow{P}(X)(a, b)$!

Two paths are **dihomotopic** if there is a dihomotopy between them, or equivalently, if they are in the same path-connected components of $\overrightarrow{P}(X)(a, b)$.

Example I

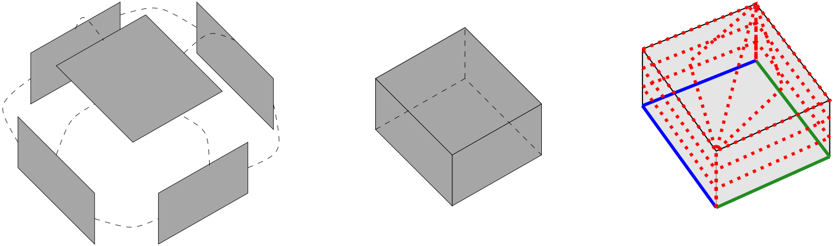


dihomotopic



non-dihomotopic

Example II : Fahrenberg's matchbox



A category of dipaths ?

Can we form the following category ?

- objects are points,
- morphisms are dipaths,
- identities are constant paths,
- composition is concatenation.

A category of dipaths ?

Can we form the following category ?

- objects are points,
- morphisms are dipaths,
- identities are constant paths,
- composition is concatenation.

Answer : No, the concatenation is not associative...

A category of dipaths ?

...but we can form the fundamental category $\overrightarrow{\pi_1}$:

- objects are points,
- morphisms are dipaths modulo dihomotopy,
- identities are dihomotopy classes of constant paths,
- composition is concatenation modulo dihomotopy.

because concatenation is associative modulo dihomotopy.

A category of traces

Actually, concatenation is associative modulo reparametrization
γ reparametrizes to ρ if there is a surjective, monotonic and continuous function $r : [0,1] \longrightarrow [0,1]$ such that $\rho = \gamma \circ r$. We call **trace** of a dipath γ and note $\langle\gamma\rangle$, the equivalence class of γ modulo reparametrization.

We can form the **category of traces** $\overrightarrow{T}(X)$:

- objects are points,
- morphisms are traces,
- identities are traces of constant paths,
- composition is concatenation modulo reparametrization.

We can also define the **trace space** $\overrightarrow{T}(X)(a,b)$ as the quotient space of $\overrightarrow{P}(X)(a,b)$ modulo reparametrization.

Objective

- study those concurrent systems through their geometry (dipaths, traces, dihomotopies)

- homology = essential notion, computable abstraction of homotopy
  - $\Rightarrow$ defining a directed homology
  - $\Rightarrow$ proving classical properties of this homology

Directed Homology

Related work

Candidates of directed homology :

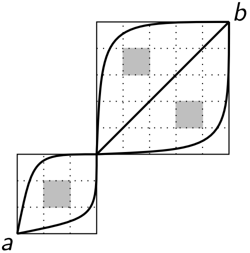
- past and future homologies **[Goubault 95]**
- ordered homology groups **[Grandis 04]**
- directed homology via ω-categories **[Fahrenberg 04]**
- homology graph **[Kahl 13]**

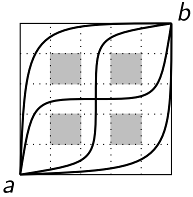
Not fine enough : do not distinguish Fahrenberg's matchbox from a point

A first idea

***Not_so_good* directed homology :**

Not_so_good(X) = classical homology of $T(X)(a, b)$

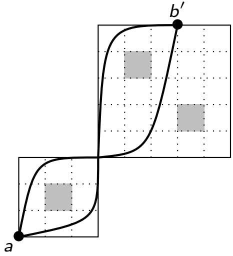


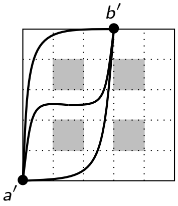


$$T(A)(a, b) \simeq 6 \textit{ point space} \simeq T(B)(a, b)$$

$$\textit{Not\_so\_good}(A) \simeq \mathcal{R}^6 \simeq \textit{Not\_so\_good}(B)$$

A first (not so) bad idea





make a, b vary

$T(A)(a, b') \simeq 4$ *point space*

$Not\_so\_good(A) \simeq \mathcal{R}^4$

no a', b' such that

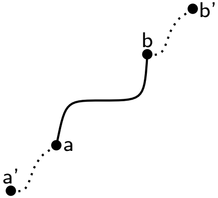
$T(B)(a', b') \simeq 4$ *point space*

$Not\_so\_good(B) \simeq \mathcal{R}^4$

Natural homology

$\mathcal{F}_X$ = category whose :

- objects are traces
- morphisms are extensions



Natural homology :

functor $\overrightarrow{H}_n(X) : \mathcal{F}_X \longrightarrow \mathbf{Mod}(\mathcal{R})$

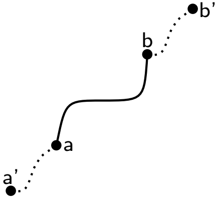
$$(a \xrightarrow{\gamma} b) \longmapsto H_{n-1}(T(X)(a,b))$$ (H_{n-1} = classical singular homology)

- $\mathcal{F}_X$ = category of factorizations **[Mac Lane 71]**
- $\overrightarrow{H}_n(X)$ = natural system **[Leech 73, Baues, Wirsching 85]**

Natural homotopy

$\mathcal{F}_X$ = category whose :

- objects are traces
- morphisms are extensions



Natural homotopy :

functor $\overrightarrow{\pi}_n(X) : \mathcal{F}_X \longrightarrow \mathbf{Set}, \mathbf{Gr}, \mathbf{Ab}$

$$(a \xrightarrow{\gamma} b) \longmapsto \pi_{n-1}(T(X)(a,b), \gamma)$$

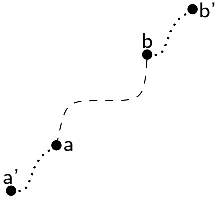
(π_{n-1} = classical homotopy)

- $\mathcal{F}_X$ = category of factorizations **[Mac Lane 71]**
- $\overrightarrow{\pi}_n(X)$ = natural system **[Leech 73, Baues, Wirsching 85]**

Bimodule homology

$\mathcal{E}_X$ = category whose :

- pairs of points (a, b), s.t. $\exists$ a dipaths from a to b
- morphisms are extensions



Bimodule homology :

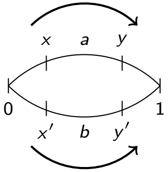
functor $\overrightarrow{H'}_n(X) : \mathcal{E}_X \longrightarrow \textbf{Mod}(\mathcal{R})$

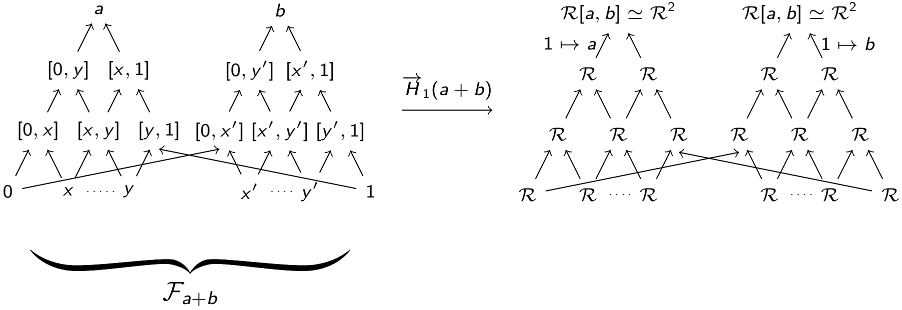
$$(a, b) \longmapsto H_{n-1}(T(X)(a, b))$$

(H_{n-1} = classical singular homology)

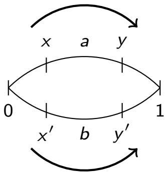
- $\mathcal{E}_X$ = enveloping category
- $\overrightarrow{H'}_n(X)$ = bimodule **[Mitchell 72]**

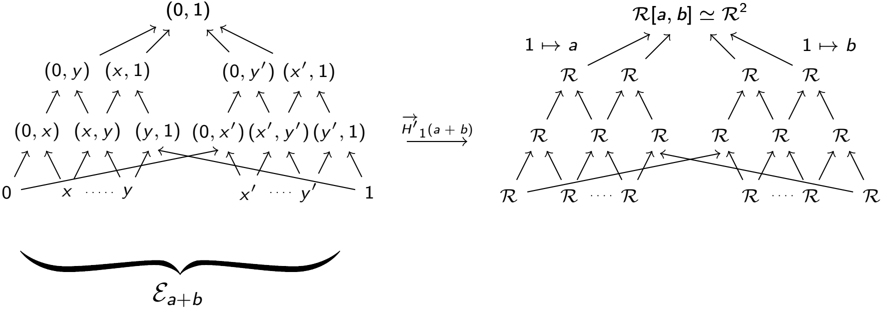
Example : first natural homology of $a + b$



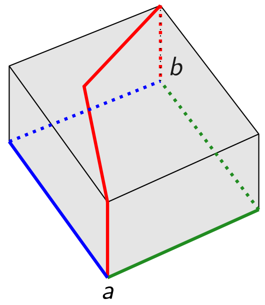


Example : first bimodule homology of $a + b$





Natural homology on Fahrenberg's matchbox



2 dipaths non dihomotopic

$\Rightarrow$ $T(X)(a, b) \simeq$ 2 point space

$\Rightarrow$ $H_0(T(X)(a, b)) \simeq \mathcal{R}^2$

$\Rightarrow$ $\overrightarrow{H}_1(X)$ not trivial

$\Rightarrow$ **natural homology detects failure of dihomotopy in Fahrenberg's matchbox**

Comparison of diagrams

Category of diagrams $\mathbf{Diag}(\mathcal{M})$

Fix a category $\mathcal{M}$, typically $\mathbf{Mod}(\mathcal{R})$, **Set**, ...
A **diagram in** $\mathcal{M}$ is a functor from any small category to $\mathcal{M}$.

Ex : $\overrightarrow{H}_n(X)$, $\overrightarrow{H'}_n(X)$, $\overrightarrow{\pi}_n(X)$ are diagrams

A morphism of diagrams from $F : \mathcal{C} \longrightarrow \mathcal{M}$ to $G : \mathcal{D} \longrightarrow \mathcal{M}$ is a pair (Φ, σ) where :

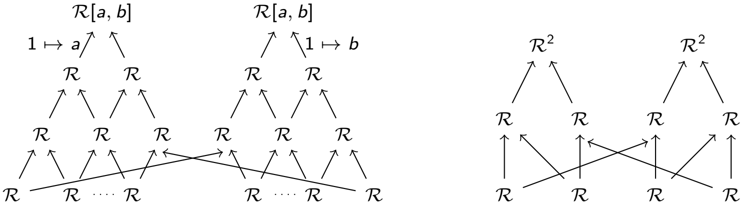
- $\Phi : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor,
- $\sigma : F \longrightarrow G \circ \Phi$ is a natural transformation.

Isos : Φ isofunctor, σ natural iso

$\overrightarrow{H}_n$, $\overrightarrow{H'}_n$, $\overrightarrow{\pi}_n$ are functors from **dTop** to $\mathbf{Diag}(\mathcal{M})$

How to compare natural homologies ?

$\overrightarrow{H}_n(A) = \overrightarrow{H}_n(B) \Rightarrow A = B$, modulo isomorphism



Crucial idea :

Compare natural homologies up-to evolutions of homology of trace spaces with time.
Idea similar to bisimulations in concurrent systems.

Bisimulations of diagrams

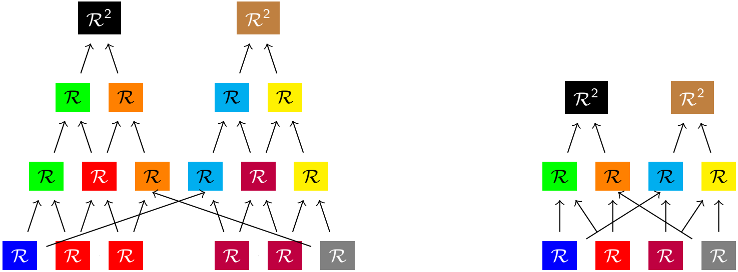
Based on the general theory of bisimulations of **[Joyal et al. 94]**

An **open map** is a morphism of diagrams (Φ, σ) from $F : \mathcal{C} \longrightarrow \mathcal{M}$ to $G : \mathcal{D} \longrightarrow \mathcal{M}$ such that :

- σ is a natural iso,
- Φ is surjective on objects,
- Φ is fibrational in the following sense : for every morphism of the form $j : F(c) \longrightarrow d'$ in $\mathcal{D}$, there is a morphism $i : c \longrightarrow c'$ in $\mathcal{C}$ with $F(i) = j$.

We say that two diagrams are **bisimilar** if there is a zigzag of open maps between them

Examples



$\overrightarrow{H}_n(X)$ is bisimilar to $\overrightarrow{H'}_n(X)$

the first natural homology of the matchbox is not bisimilar to the one of a point space

Computability

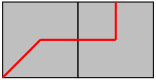
Cubical complex and discrete traces

Euclidian cubical complex : any subspace of $\mathbb{R}^n$ which is a finite union of cubes of the form

$$[a_1, a_1 + \alpha_1] \times \ldots \times [a_n, a_n + \alpha_n]$$

with $a_i \in \mathbb{Z}$ and $\alpha_i \in \{0, 1\}$.

discrete trace = trace which is a glueing of segments joining center of cubes



f_X = category of discrete traces and extensions by discrete traces

Discrete natural homology $\overrightarrow{h}_n(X)$:

functor $\overrightarrow{h}_n(X) : \boxed{f_X} \longrightarrow \mathbf{Ab}$

$$(a \xrightarrow{\gamma} b) \longmapsto H_{n-1}(T(X)(a, b))$$

Computability

Theorem :

Given an Euclidian cubical complex X, $\overrightarrow{h}_n(X)$ is :

- computable
- bisimilar to $\overrightarrow{H}_n(X)$

Proof of computability :
- compute a finite representation of trace spaces **[Raussen, Ziemianski]**,
- compute its homology.

Corollary :

Given two Euclidian cubical complexes, it is decidable wether they have the same natural homology (when computed in real numbers).

Proof :
Bisimilarity is decidable by reducing the existential theory of the reals.

Exactness Axiom

Exactness axiom in classical homology

If (A, X) is a topological pair, the relative homology $H_n(X, A)$ is the homology of the chain complex $C_n(X)/C_n(A)$

Exactness axiom :

There is a long exact sequence :

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

Exactness axiom in classical homology, general form

the sequence :

$$C(A) \xrightarrow{\quad i \quad} C(X) \xrightarrow{\quad p \quad} C(X)/C(A)$$

is short exact

Long exact sequence in homology :

If

$$A \xrightarrow{\quad i \quad} B \xrightarrow{\quad p \quad} C$$

is a short exact sequence of chain complexes, then there is a long exact sequence in modules of the form :

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(p)} H_n(C) \xrightarrow{\quad \partial_n \quad} H_{n-1}(A) \longrightarrow \cdots$$

nice for computations, more general in Abelian categories

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects,
- kernels,
- images/cokernels,
- subquotients (exactness of some morphisms),

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels,
- images/cokernels,
- subquotients (exactness of some morphisms),

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels,
- subquotients (exactness of some morphisms),

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels, OK, but more complicated,
- subquotients (exactness of some morphisms),

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels, OK, but more complicated,
- subquotients (exactness of some morphisms), OK.

Exactness in diagrams ?

Diag(Mod($\mathcal{R}$)) is not Abelian

Which ingredients ?

- zero objects, no zero objects, but null objects (diagrams with values 0),
- kernels, OK defined levelwise,
- images/cokernels, OK, but more complicated,
- subquotients (exactness of some morphisms), OK.

Theorem :

Diag(Mod($\mathcal{R}$)) is a homological category in the sense of **[Grandis 91]**.

Almost exactness in diagrams

Theorem [Grandis 91] :

Let $\mathcal{A}$ be a homological category.
For every short exact sequence in $C_\bullet(\mathcal{A})$:

$$\mathrm{U} \xrightarrow{\mathrm{m}} \mathrm{V} \xrightarrow{\mathrm{p}} \mathrm{W}$$

there exists a long sequence of order two in $\mathcal{A}$:

$$\cdots \longrightarrow H_n(V) \xrightarrow{H_n(p)} H_n(W) \xrightarrow{\partial_n} H_{n-1}(U) \xrightarrow{H_{n-1}(m)} H_{n-1}(V) \longrightarrow \cdots$$

natural in the short exact sequence.
Moreover, there are conditions to turn the long sequence to an exact sequence. In particular, $\mathcal{A}$ is modular iff this sequence is always exact.

Bad news : $\mathbf{Diag(Mod(}\mathcal{R}\mathbf{))}$ is not modular

Homotopy Axioms

Homotopy axioms for classical homology

Homotopy axiom, original form

If $f, g : X \longrightarrow Y$ are homotopic, then $H_n(f) = H_n(g)$.

Homotopy axiom, v.1

If $f : X \longrightarrow Y$ is a homotopy equivalence, then $H_n(f) : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism.

Homotopy axiom, v.2

If $f : X \longrightarrow Y$ is a weak homotopy equivalence, then $H_n(f) : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism.

Dihomotopy axioms for natural homology, v.2

For free :

Dihomotopy axiom, v.2.0

Let $f : X \longrightarrow Y$ be a dimap. If for every n,

$$\overrightarrow{\pi}_n(f) : \overrightarrow{\pi}_n(X) \longrightarrow \overrightarrow{\pi}_n(Y)$$

is a isomorphism of diagrams, then for every n,

$$\overrightarrow{H}_n(f) : \overrightarrow{H}_n(X) \longrightarrow \overrightarrow{H}_n(Y)$$

is a isomorphism of diagrams.

Proof :
Apply the homotopy axiom, form v.2 on trace spaces.

Dihomotopy axioms for natural homology, form 2

Better, (almost) as free as v.2.0 :

Dihomotopy axiom, v.2.1

Let $f : X \longrightarrow Y$ be a dimap. If for every n,

$$\overrightarrow{\pi}_n(f) : \overrightarrow{\pi}_n(X) \longrightarrow \overrightarrow{\pi}_n(Y)$$

is a open map, then for every n,

$$\overrightarrow{H}_n(f) : \overrightarrow{H}_n(X) \longrightarrow \overrightarrow{H}_n(Y)$$

and

$$\overrightarrow{H'}_n(f) : \overrightarrow{H'}_n(X) \longrightarrow \overrightarrow{H'}_n(Y)$$

are open maps.

Proof :
Apply the homotopy axiom, form 2 on trace spaces + reasoning on open maps.

Which dihomotopy equivalences for v.1 ?

A **future deformation retract** of X on a sub-dspace A is a continuous map

$$H : X \longrightarrow \mathfrak{I}(X) \subseteq \overrightarrow{P}(X)$$

such that :

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap ;
- for every δ of A from z to $H_1(x)$ there is a dipath γ of X from y to x with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to δ.

Inessential equivalence :

Two dspaces are inessentially equivalent iff there is a zigzag of future and past deformation retracts between them.

Why not $\overrightarrow{P}(X)$?

In classical algebraic topology :
if $f : X \longrightarrow Y$ is a homotopy equivalence, $P(x, y)$ and $P(f(x), f(y))$ are homotopically equivalent because paths induces homotopy equivalence by concatenation :

$$\gamma \star \_ : P(X)(z, x) \longrightarrow P(X)(z, y) \quad \delta \mapsto \gamma \star \delta$$

is a homotopy equivalence.

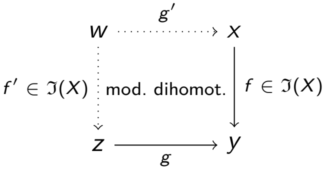
In directed algebraic topology : dipaths do not have this property

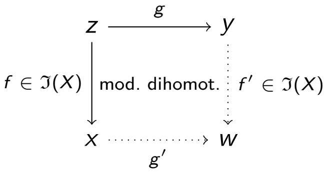
Inessential dipaths

Idea from **[Fajstrup, Goubault, Haucourt, Raussen]** for category of components.

The set $\mathfrak{I}(X)$ of inessential dipaths of X is the largest set of dipaths such that :

- for every $\gamma \in \mathfrak{I}(X)$ from x to y, for every $z \in X$ such that $\overrightarrow{P}(X)(z,x) \neq \varnothing$, the map $\gamma \star \_ : \overrightarrow{P}(X)(z,x) \longrightarrow \overrightarrow{P}(X)(z,y) \;\; \delta \mapsto \gamma \star \delta$ is a homotopy equivalence ;
- symmetrically for $\_ \star \gamma$;
- $\mathfrak{I}(X)$ has the right and left Ore condition modulo dihomotopy :





Why inessential equivalence ?

- classify as we expect many dspaces (for example, distinguish the matchbox and the point),
- in-between Grandis' reversible and dihomotopy equivalences,
- its action on the fundamental category corresponds to the category of components,
- because of the preservation of the homotopy type of the space of dipaths, it has a deep relation with $(\infty, 1)$-categories (directed homotopy hypothesis).

Dihomotopy axiom, v.1

Dihomotopy axiom v.1

If two **Euclidian cubical complexes** X and Y are inesentially equivalent then $\overrightarrow{H}_n(X)$, $\overrightarrow{H'}_n(X)$, $\overrightarrow{H}_n(Y)$ and $\overrightarrow{H'}_n(Y)$ are bisimilar.

Conclusion : natural and bimodule homologies are invariant of inessential equivalence, at least on Euclidian cubical complexes, where we can do computations.