Fibrational Bisimulations and Quantitative Reasoning: Extended Version

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Abstract

Bisimulation and *bisimilarity* are fundamental notions in comparing state-based systems. Their extensions to a variety of systems have been actively pursued in recent years, a notable direction being *quantitative* extensions. In this paper we enhance a categorical framework for such extended (bi)simulation notions. We use coalgebras as system models and fibrations for organizing predicates—following the seminal work by Hermida and Jacobs. Endofunctor liftings are crucial predicate-forming ingredients; the first contribution of this work is to extend several extant lifting techniques from particular fibrations to $\mathbf{CLat}_\wedge$-fibrations over $\mathbf{Set}$. The second contribution of this work is to introduce *endolifting morphisms* as a mechanism for comparing predicates between fibrations. We apply these techniques by deriving some known properties of the Hausdorff pseudometric and approximate bisimulation in control theory.

Keywords: fibration; approximate bisimulation; behavioural metric; and functor lifting

1 Introduction

In the study of transition systems, *bisimulation relations* are a fundamental concept, and their categorical study revealed the importance of *coalgebras*. One approach to characterize bisimilarity is via *liftings* of the coalgebra functor along *fibrations* [17], which are a well-established framework to attach relational structures on categories for modeling transition systems and programming languages [19].

Recently, there is emerging interest in quantitative analysis of transition systems. *Behavioural metrics* were introduced in [10, 29] to refine bisimilarity for probabilistic transition systems. Metrics give a real number for each pair of states in a transition system, while a relation can only provide a bit for each pair. Therefore a metric can indicate a degree to which the behaviour of two states differ, whereas a bisimilarity relation can only indicate whether or not those behaviours differ. From this observation, a common desideratum for behavioral metrics associated with coalgebras is that two states should have distance 0 if and only if they are bisimilar.

Bisimilarity and behavioural metrics are also analogous on a categorical level. Behavioural metrics were recently shown to be constructible from liftings of the coalgebra functor to categories of (pseudo)metrics [4, 5], similar to how Hermida-Jacobs bisimulations are constructed from liftings of a functor to the category of relations. This type of construction is known generally as a coalgebraic predicate and can be performed when a lifting of the coalgebra functor is known.

These developments present two natural issues. The first is an open-ended quest for liftings of functors in general fibrations. These liftings are the rare ingredients in forming coalgebraic predicates, so having more liftings in more fibrations allows us to express more coalgebraic predicates. The second issue is more recent and concerns the desired relationship between behavioural metrics and bisimilarity mentioned above. Given some liftings in different fibrations, is there a relationship between the liftings we can use to verify a relationship between the coalgebraic predicates they define on a given coalgebra?

The main purpose of this paper is to provide solutions to these issues within the context of $\mathbf{CLat}_\wedge$*-fibrations over* $\mathbf{Set}$; technically speaking, they are the fibrations where fiber categories are complete lattices and reindexing functors preserve all meets. This class of fibrations is a convenient setting for the general theory of bisimulation and bisimilarity. It contains forgetful functors from many well-known categories, such as categories of topological spaces, pseudometric spaces, binary relations, quantale-enriched small categories, and so on. The main contributions of this paper pertain to these two issues:

- Regarding the first issue, we propose several methods to lift functors along $\mathbf{CLat}_\wedge$-fibrations over $\mathbf{Set}$. The lifting method we introduce generalize several existing constructions. The first is the *codensity lifting of endofunctors*, generalizing Baldan et al.'s *Kantorovich lifting* [5] to arbitrary $\mathbf{CLat}_\wedge$-fibrations over $\mathbf{Set}$. This lifting also represents a further development of the codensity lifting of *monads* [21]. The second is the construction of an *enriched left Kan extension* using the *canonical symmetric monoidal closed structure* [23] on the total category of $\mathbf{CLat}_\wedge$-fibrations over $\mathbf{Set}$. This generalizes Balan et al.'s construction [4] of enriched left Kan extension for quantale-enriched small categories. The third is a reformulation and generalization of Bonchi et al.'s *Wasserstein lifting* [6] using Hermida's adjoint lifting theorem for opfibrations [16]. We also introduce the bifibration of liftings of endofunctors, in which we can manipulate liftings. As an example, we construct the *Hausdorff metric* as the pushforward of the lifting of the list functor along a particular natural transformation.

- Regarding the second issue, we propose a concept of *morphism between liftings*. We use them to provide facilities for establishing relationships between the coalgebraic predicates provided by these liftings on coalgebras. We illustrate the utility of this approach with several examples. First, we show that the specialization preorder construction translates bisimilarity topologies on coalgebras of polynomial functors to bisimilarity relations. Second, we translate metrics to relations to show the kernels of many behavioural metrics are bisimilarity relations. Third, we demonstrate the translation of *approximate functions* to *ϵ-approximate relations* [14], which is the key technical tool used in control theory.

Outline. In Section 2, we sketch the important technical background for this work, including partial order fibrations (over **Set**) and Hermida and Jacobs' theory of coalgebraic bisimulations in fibrations. As mentioned above, the theory requires a lifting of a functor, and the general question is the construction of liftings, which we address in Section 5. In Section 3, we concretely illustrate the constructions of topological and relational bisimilarities and their comparison using the specialization order functor. In Section 4 we survey the properties of $\mathbf{CLat}_\wedge$-fibration over **Set** in detail. In Section 5 we present generalizations of several extant techniques for producing liftings in particular fibrations to our more general class of $\mathbf{CLat}_\wedge$-fibrations over **Set**. Finally, in Section 6, we use so-called morphisms between endoliftings to establish relationships between coalgebraic predicates, focusing how topological / metric bisimilarity induces bisimilarity relations. We also illustrate an example of this story from control theory: deriving approximate functions from ϵ-approximate relations and deriving bisimilarity as the kernel of behavioural metrics.

This is an extended version of [28]; many proofs not present in the conference version have been added. We have added (Example 4.3) a description of the symmetric monoidal closed structure on Q-**Cat**, (Section 5.4) a reformulation of Bonchi et al's Wasserstein lifting [6] in $\mathbf{CLat}_\wedge$-fibrations and (Theorem 4.3) a characterization of fibered functors between $\mathbf{CLat}_\wedge$-fibrations preserving fibered meets. A major new example includes a class of liftings of functors to **Top**, which we show create special topologies on coalgebras with the property that two points are topologically indistinguishable if and only if they are bisimilar. We manage to prove this property with the aid of morphisms between liftings. We also fix the incorrect definition of the tensor unit of the symmetric monoidal closed structure (Section 4.2) in the conference version.

2 Background on Fibrational Coalgebra

In this paper, we are interested in *creating* and *comparing* mathematical structures (e.g., relations, topologies, pseudometrics) defined on different kinds of transition systems (e.g., nondeterministic, probabilistic, weighted, transducers) with different properties (e.g., bisimilarities, language inclusions, behavioural metrics).

We capture these three different components with three largely orthogonal categorical abstractions. First, we use *coalgebras* as a means of modeling many kinds of transition systems. Second, *fibrations* define the type of mathematical structures we can create on the set of states of a coalgebra. Third, *functor liftings* create particular instances in the fibration with different properties. We review each of these categorical concepts in this section. We assume familiarity with basic category theory, but not necessarily with the theory of fibrations.

2.1 Coalgebras

Coalgebra is our tool of choice for modeling state-based transition systems. Given an endofunctor F on **Set**, an F*-coalgebra* in **Set** is a pair (I, f) consisting of a set I and a function $f : I \to FI$. The set is often called the *carrier* of the coalgebra, while the function provides the *transition structure* of the coalgebra.

This pair is usually interpreted as a transition system under the following scheme. The (object part of the) functor F is thought of as an operation which sends a set of states to the set of all possible transition structures on that set. The set I is the set of states of a transition system. Then FI is the set of all possible transition structures available on this set of states I, so the transition structure map $f : I \to FI$ assigns one of these possible transition structures to every state in I.

A *coalgebra morphism* $h : (I, f) \to (J, g)$ is a function on the underlying state sets $h : I \to J$ which respects the transitions in the source coalgebra, meaning $g \circ h = Fh \circ f$. F-coalgebras together with their morphisms form a category we denote by $\mathbf{Coalg}(F)$.

By varying the functor F, we can capture a wide variety of transition system types, including deterministic and nondeterministic finite automata, Mealy and Moore machines, probabilistic transition systems, Markov decision processes, Segala systems and many more. For more background on the theory of coalgebra, we recommend consulting [27].

2.2 Fibrations

A *fibration* over a category $\mathbb{B}$ is a functor $\pi : \mathbb{E} \to \mathbb{B}$ with the *cartesian lifting property*, which we describe later. The source category of the fibration, $\mathbb{E}$, is referred to as the *total category* and the target, $\mathbb{B}$, is the *base category*. In this work, we usually consider fibrations over **Set** since we build extra structures on **Set**-based coalgebras. Indeed, the total categories we are most interested in are sets equipped with some extra structure, such as sets with relations or sets with metrics. In many of these cases, the forgetful functor is a fibration.

Example 2.1. *The forgetful functors from the following categories to* **Set** *are fibrations:*

- **Pre** *is the category of preorders and monotone functions between them.*

- **ERel** *is the category of endorelations. An object is a pair* (I, R) *of a set* I *and a relation* $R \subseteq I \times I$*. Morphisms are functions which preserve the relation, meaning* $f : (I, R) \to (J, S)$ *is a function* $f : I \to J$ *such that* $(i, i') \in R$ *implies* $(f(i), f(i')) \in S$*.*

- *Objects of* **BVal** *are pairs* (I, r) *of a set* I *and a function* $r : I \times I \to \mathbb{R}^+$ *with no constraints. We regard* r *as the most relaxed form of metrics on* I*. Morphisms in this category are required to be non-expansive, so* $f : (I, r) \to (J, s)$ *satisfies* $s(f(i), f(i')) \leq r(i, i')$ *for all* $i, i' \in I$*.*

- **Top** *is the category of topological spaces and continuous functions between them.*

In this article, we write $\pi_{\mathbb{C}}$ *to mean the forgetful functors from those categories* ($\mathbb{C} = \mathbf{Pre}, \mathbf{ERel}, \mathbf{BVal}, \mathbf{Top}$)*. We also use this notation to refer to the forgetful functors from those of Example 4.1.*

The total category of a fibration is often depicted vertically above the base category and language referencing this physical configuration is common. Saying an object or a morphism e in $\mathbb{E}$ is *above* an object or a morphism b in $\mathbb{B}$ means $\pi e = b$.

The collection of objects and morphisms above an object I and the identity morphism id_I, respectively, is called the *fiber over* I. Each of these fibers is itself a subcategory of $\mathbb{E}$, denoted by $\mathbb{E}_I$. For example, $\mathbf{ERel}_I$ is the category of relations on I with a morphism from (I, R) to (I, S) if $(i, i') \in R$ implies $(i, i') \in S$ (that is, $R \subseteq S$). Hence, $\mathbf{ERel}_I$ is a thin category isomorphic to the complete lattice of relations on I.

Each of these examples has a peculiar feature. In the fibration **ERel**, given a function $f : I \to J$ and a relation $S \subseteq J \times J$ on J, there is a largest relation R on I so that f becomes a relation-preserving function (i.e., a morphism in **ERel**): $(i, i') \in R$ if and only if $(f(i), f(i')) \in S$. In **Top**, given a topology on J, the coarsest topology on I making f a continuous function fulfills a similar role. The *cartesian lifting property* is a categorification of these observations.

First, we introduce the concept of *cartesian morphisms*. Let $\pi : \mathbb{E} \to \mathbb{B}$ be a functor. For objects $X, Y \in \mathbb{E}$ and a morphism $f : \pi X \to \pi Y$ in $\mathbb{B}$, by $\mathbb{E}_f(X, Y)$ we mean the set $\{\dot{f} \in \mathbb{E}(X, Y) \mid \dot{f} \text{ is above } f\}$. A morphism $\dot{f} : X \to Y$ in $\mathbb{E}$ is called *cartesian* if for any $Z \in \mathbb{E}$ and $g : \pi Z \to \pi X$, the following postcomposition function is invertible:

$$\dot{f} \circ - : \mathbb{E}_g(Z, X) \to \mathbb{E}_{\pi \dot{f} \circ g}(Z, Y).$$

Equivalently, $\dot{f}$ satisfies the following universal property: for any $\dot{h} : Z \to Y$ above $\pi \dot{f} \circ g$, there exists a unique $\dot{g} : Z \to X$ above g such that $\dot{f} \circ \dot{g} = \dot{h}$.

Cartesian morphisms in **ERel** preserve *and reflect* their source relation, and in **BVal** they are isometries, replacing the inequality in the condition for non-expansiveness with equality.

We then say that $\pi : \mathbb{E} \to \mathbb{B}$ is a *fibration* if it satisfies the following *cartesian lifting property*: for every morphism $f : I \to J$ in $\mathbb{B}$ and every object Y in $\mathbb{E}$ above J, there is a cartesian morphism $\dot{f}$ into Y above f. The morphism $\dot{f}$ is called the *cartesian lifting of f with Y*. We assume that we have chosen a cartesian lifting for each pair (f, Y), and denote it by $\overline{f}(Y) : f^*Y \to Y$.

For a $\mathbb{B}$-morphism $f : I \to J$, the assignment $Y \mapsto f^*Y$ becomes a functor of type $\mathbb{E}_J \to \mathbb{E}_I$. This is called the *pullback (functor)* along f.[1] Moreover the assignment $f \mapsto f^*$ is functorial up-to natural isomorphisms $(g \circ f)^* \cong f^* \circ g^*$ and $\mathrm{id}_I^* \cong \mathrm{Id}_{\mathbb{E}_I}$. When these isomorphisms are identities, the fibration is called *split*.

A *partial order fibration* is a fibration where each fiber category $\mathbb{E}_I$ is a partial order. They are split and always faithful as functors. We introduce a notation in a partial order fibration $\pi : \mathbb{E} \to \mathbb{B}$: for objects $X, Y \in \mathbb{E}$ and a $\mathbb{B}$-morphism $f : \pi X \to \pi Y$, by $f : X \dot{\to} Y$ we mean the sentence: "there exists a (necessarily unique) $\mathbb{E}$-morphism $\dot{f} : X \to Y$ such that $\pi \dot{f} = f$".

2.3 Liftings

Another major object of study in this work are liftings of a functor. Given an endofunctor F on **Set** and two fibrations $\pi : \mathbb{E} \to \textbf{Set}$ and $\rho : \mathbb{F} \to \textbf{Set}$, a *lifting* of F is a functor $\dot{F} : \mathbb{E} \to \mathbb{F}$ such that $\rho \circ \dot{F} = F \circ \pi$.

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\dot{F}"] \arrow[d, "\pi"'] & \mathbb{F} \arrow[d, "\rho"] \\ \textbf{Set} \arrow[r, "F"'] & \textbf{Set} \end{tikzcd}$$

Such a lifting can be restricted to the functor $\dot{F}|_I : \mathbb{E}_I \to \mathbb{F}_{FI}$ between fibers over any set I. In many of the cases we consider $\pi = \rho$. To emphasize this particular situation we will call such an $\dot{F}$ an *endolifting* of F along π. When the fibration can be inferred, we simply say that $\dot{F}$ is an $\mathbb{E}$ lifting of F. In [15], endoliftings were called *modalities*.

Fibrational pullback and endoliftings in the total category are used to create "coalgebraic predicates" in [15] on individual coalgebras in a process we describe in the next section. These are abstract predicates which generalize both classical predicates and relations in the context of a categorical logic based on fibrations, see [19]. However, we will call these structures "coinductive invariants" to avoid the implication they are sets of states.

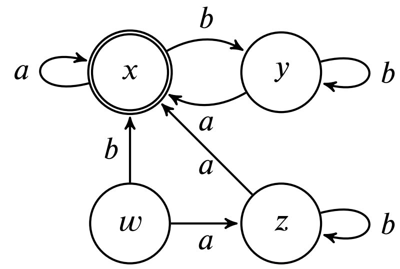
3 Motivation: Coinductive Relations and Topologies

In this section, we concretely illustrate the *creation* of coinductive invariants and an example of a *comparison* we would like to make between two such invariants. As a running example, we consider a particular coalgebra for the functor $FX = 2 \times X \times X$. For convenience, we name the elements of $2 = \{\top, \bot\}$.

[1] In this paper we shall use the word *pullback* in this fibrational sense. This usage generalizes the word's common meaning as a limit of a cospan in a category. Specifically, the latter gives a (fibrational) pullback in a codomain fibration. See [19].

Example 3.1. *Let $I = \{x, y, z, w\}$ and $\delta : I \to 2 \times I \times I$ be defined by $\delta(x) = (\top, x, y)$, $\delta(y) = (\bot, x, y)$, $\delta(z) = (\bot, x, z)$ and $\delta(w) = (\bot, z, x)$.*

This coalgebra represents a DFA on the alphabet $\{a, b\}$ where $\top$ / $\bot$ gives acceptance / rejection to a state, the second component of the transition structure names the next state after reading a, and the third component gives the state after reading b.



3.1 Creating Coinductive Relations

Creating coinductive invariants requires two basic ingredients: a pullback in a fibration and an endolifting of the coalgebra functor. In the case of the fibration **ERel**, these have two core roles:

- Given a function $f : X \to Y$ and relation $S \subseteq Y \times Y$ on Y, the relation pullback produces a relation on X, namely f^*S defined by $(x, x') \in f^*S$ if and only if $(f(x), f(x')) \in S$. In particular, the relation pullback along a coalgebra structure $f : X \to 2 \times X \times X$ takes a relation on $2 \times X \times X$ and produces a relation on X.
- Given a relation R on X, a relation lifting produces a relation on the set $2 \times X \times X$.

The basic idea of Hermida and Jacobs [17] is to refine the complete relation on the state space by alternating these two operations. In pseudocode, this looks like Algorithm 1. This sequence eventually reaches a fixed point (perhaps requiring transfinitely many steps) by monotonicity arguments we will cover later.

Algorithm 1 Pseudocode for the creation of a coinductive invariant

Input: A coalgebra $\delta : I \to 2 \times I \times I$, and a relation lifting of F.
Output: The coinductive invariant of δ created by the lifting.

$R_0 = I \times I$; $S_0 = \text{lifting}(R_0)$; i = 0;
do
  i = i+1;
  $R_i = \delta^* S_{i-1}$; ▷ pullback
  $S_i = \text{lifting}(R_i)$;
while $R_i \neq R_{i-1}$
return R_i;

Perhaps the most famous example of this process is the definition of bisimilarity due to Hermida and Jacobs [17]. This uses the "canonical relation lifting", see Definition 5.1

Definition 3.1. *The* canonical relation lifting *of $FX = 2 \times X \times X$ is a functor $Rel(F)$* : **ERel** $\to$ **ERel** *defined by $Rel(F)(X, R) = (2 \times X \times X, \Delta_2 \times R \times R)$ on objects and by $Rel(F)(f) = (\mathrm{id}_2, f, f)$ on morphisms. Here $\Delta_2 = \{(\top, \top), (\bot, \bot)\}$ is the diagonal relation on 2.*

Example 3.2. *We compute the coinductive invariant created by the canonical relation lifting on the coalgebra of Example 3.1. This relation turns out to be the bisimilarity relation of this coalgebra.*

We start with $R_0 = \{(i, j) \mid i, j \in I\}$. Immediately, $S_0 = \Delta_2 \times R_0 \times R_0$.

Next, $R_1 = \delta^* S_0 = \{(x, x)\} \cup \{(i, j) \mid i, j \in \{y, z, w\}\}$, since neither $(\top, \bot)$ nor $(\bot, \top)$ are in Δ_2. The relation lifting then gives $S_1 = \Delta_2 \times R_1 \times R_1$.

In the next iteration of the loop, we get $R_2 = \delta^* S_1 = \{(x, x)\} \cup \{(w, w)\} \cup \{(i, j) \mid i, j \in \{y, z\}\}$, further refining R_1. In R_2, x is related only to itself for the same reason as in R_1, but w is also *related only to itself since the second component of its transition (z) is not related to the second component of y or z's transition (x) in R_1.*

R_3 is exactly R_2, so R_2 is the coinductive invariant generated by this relation lifting, and indeed is the bisimilarity relation on this coalgebra.

If we change the relation lifting, we get a different coinductive invariant. As an example, consider the following relation lifting:

Definition 3.2. *The* language inclusion lifting *of $FX = 2 \times X \times X$ is a functor $Li(F)$* : **ERel** $\to$ **ERel** *defined by $Li(F)(X, R) = (2 \times X \times X, \leq_2 \times R \times R)$ where $\leq_2 = \{(\bot, \bot), (\bot, \top), (\top, \top)\}$.*

The coinductive invariant created by this lifting is the language inclusion relation. We illustrate this by computing it for our example coalgebra.

Example 3.3. *We compute the coinductive invariant created by the language inclusion lifting on the coalgebra of Example 3.1. Just as in Example 3.2, $R_0 = \{(i, j) \mid i, j \in I\}$ and $S_0 = \leq_2 \times R_0 \times R_0$.*

Next, $R_1 = \delta^* S_0 = R_0 \setminus \{(x, y), (x, z), (x, w)\}$ since $\top \not\leq_2 \bot$.

To compute R_2, we note that $(x, z) \notin R_1$, so since $\delta(w) = (\bot, z, x)$ and $\delta(z) = (\bot, x, z)$, we must have $(\delta(z), \delta(w)), (\delta(w), \delta(z)) \notin S_1$. Similar reasoning shows w is not related to y in either order under R_2. Further, $(\delta(y), \delta(x)), (\delta(z), \delta(x)) \in S_1$ but not conversely, and that $(\delta(y), \delta(z)), (\delta(z), \delta(y)) \in S_1$. Hence $R_2 = \Delta_I \cup \{(y, x), (z, x), (y, z), (z, y)\}$.

It is straightforward to check that $R_3 = R_2$, so R_2 is the coinductive invariant created by this lifting.

Notice that this relation is indeed the language inclusion relation for this automaton: x accepts the language $\{\epsilon\} + \{a, b\}^* a$, both y and z accept the language $\{a, b\}^* a$ and w accepts the language $\{b\} + \{a, b\}^+ a$.

3.2 Creating Coinductive Topologies

We can use essentially the same components in the fibration **Top** of topological spaces to create coinductive invariants in this fibration.

- The pullback operation in the fibration $\pi_{\mathbf{Top}} : \mathbf{Top} \to \mathbf{Set}$ is given by the initial topology. Given a topological space (Y, σ) and a function $f : X \to Y$, the *initial topology* $f^*(Y, \sigma)$ (see e.g. [1, Example 8.8(2)]) is the coarsest topology on X such that f becomes a continuous function of type $f^*(Y, \sigma) \to (Y, \sigma)$. Explicitly, it is given by $f^*(Y, \sigma) \triangleq (X, \{f^{-1}[O] \mid O \in \sigma\})$.
- Given a topology τ on X, a topological lifting for the functor F produces a topology on the set $2 \times X \times X$. There are multiple possible topological liftings for a functor.

By starting with the coarsest possible topology on a state space X, the indiscrete topology $\{\emptyset, X\}$, and alternating applications of lifting and pullback until we reach a fixpoint, we create topologies on coalgebras.

Definition 3.3. *The* Sierpiński topology lifting *of $FX = 2 \times X \times X$ is a functor $Si(F)$:* $\mathbf{Top} \to \mathbf{Top}$ *taking (X, τ) to $(2 \times X \times X, s \times \tau \times \tau)$, where $s = \{\emptyset, \{\top\}, 2\}$ is the Sierpiński topology on 2 and $\times$ in the second component denotes the product topology.*

Example 3.4. *We compute the coinductive invariant created on the coalgebra of Example 3.1 by the Sierpiński topology lifting. In this computation, we will denote topologies on the state space by τ_i and topologies on the transition space by σ_i.*

We start with $\tau_0 = \{\emptyset, I\}$, which means $\sigma_0 = \{\emptyset, \{\top\} \times I \times I, 2 \times I \times I\}$.

The δ-preimages of the sets in σ_0 form $\tau_1 = \{\emptyset, \{x\}, I\}$. Therefore, a basis for σ_1 consists of nine sets—the empty set and all eight combinations of the two nonempty sets in each of the three topologies in the product. Note that unlike σ_0, this basis is not union-closed.

Though writing out σ_1 in full is distractingly large, we can still compute τ_2 relatively easily since the preimages of a basis form a basis of the initial topology. All basis elements with $\{\top\}$ in their first component have preimage $\{x\}$, so we need only consider basis elements starting with 2. The preimage of $2 \times \{x\} \times I$ is $\{x, y, z\}$, the preimage of $2 \times I \times \{x\}$ is $\{w\}$, and the remaining basis elements yield sets in τ_1. Thus, $\tau_2 = \{\emptyset, \{x\}, \{w\}, \{x, w\}, \{x, y, z\}, I\}$.

It takes much more (straightforward) work to compute τ_3, but it turns out $\tau_3 = \tau_2$.

3.3 Comparing Coinductive Invariants

Having seen the computations of three different coinductive invariants on the same coalgebra terminate at the third step, a natural question is whether there is any relationship between the topologies and relations computed in Examples 3.2, 3.3 and 3.4. In fact, there is a very close relationship via the specialization preorder, which we recall next.

Definition 3.4. *Let (X, τ) be a topological space. The* specialization preorder *on this topology, which we denote* Spec(τ)*, is the relation defined by $(x, x') \in \mathrm{Spec}(\tau)$ if and only if every open set containing x also contains x'.*

It is straightforward to check that the specialization preorder of the Sierpiński coinductive invariant computed in Example 3.4 is precisely the relation computed in Example 3.3.

Example 3.5. *In the topology τ_2 of Example 3.4, every open set containing y or z also contains x, so $(y, x), (z, x) \in \mathrm{Spec}(\tau_2)$. Similarly, every open set containing y or z contains the other, so these are equivalent in* $\mathrm{Spec}(\tau_2)$*. Finally, $\{w\}$ is itself open, so w is related only to itself in* $\mathrm{Spec}(\tau_2)$*. Hence,* $\mathrm{Spec}(\tau_2) = R_2$ *of Example 3.3.*

In fact, the eagle-eyed reader may notice that $\mathrm{Spec}(\tau_i) = R_i$ *from these examples.*

Of course, we really are interested in establishing this as a general fact, that for *all* coalgebras of the functor $FX = 2 \times X \times X$ the specialization preorder of the topology created by $Si(F)$ is exactly the relation created by $Li(F)$, language inclusion. Even better would be a proof that works for many different functors, so we could cover large families of functors in **ERel** and **Top** simultaneously. The results established in this paper, such as Proposition 6.1, provide an appropriate categorical framework in which we can use to compare many different kinds of coinductive invariants: relations and topologies via specialization preorder (Proposition 6.2), behavioural metrics and bisimilarity via kernel relations (Corollary 6.2), etc.

3.4 $\mathbf{CLat}_\wedge$-Fibrations and General Coinductive Invariants

One issue we have not confronted is the convergence of this procedure. We first note that in each of these fibrations, the fibers $\mathbf{ERel}_I$ and $\mathbf{Top}_I$ are complete lattices. The functoriality of liftings and pullbacks means they are monotone functions on these lattices. Hence, their composition is a monotone function on $\mathbb{E}_I$ and by Knaster-Tarski has a greatest fixed point, which we reach by possibly transfinitely many applications of the function from the top element in the complete lattice.

In this paper, we will therefore focus on partial order fibrations $\pi : \mathbb{E} \to \mathbf{Set}$ such that 1) each fiber category $\mathbb{E}_I$ is a complete lattice and 2) pullbacks preserve all meets in fibers. Such fibrations bijectively correspond to functors of type $\mathbf{Set}^{\mathrm{op}} \to \mathbf{CLat}_\wedge$ via the Grothendieck construction (see e.g. [19, Definition 1.10.1]), where the codomain is the category of complete lattices and meet-preserving functions between them. We call such fibrations $\mathbf{CLat}_\wedge$*-fibrations* over **Set** (see also [3, Section 4.3]), or simply $\mathbf{CLat}_\wedge$-fibrations. This is a restricted class of *topological functors* to **Set** [18], where each fiber category is a poset.

The necessary components to define coinductive invariants can be found in any combination of a $\mathbf{CLat}_\wedge$-fibration π with an endolifting $\dot{F}$ of an endofunctor F on $\mathbf{Set}$, depicted as

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\dot{F}"] \arrow[d, "\pi"'] & \mathbb{E} \arrow[d, "\pi"] \\ \mathbf{Set} \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd} \tag{1}$$

This terminology is intended to echo [15].

Definition 3.5. *In the situation* (1)*, an $\dot{F}$-*invariant *[on an F-coalgebra (I, f)] is an $\dot{F}$-coalgebra (X, α) [such that $\pi X = I$ and $\pi\alpha = f$]. An $\dot{F}$-*invariant morphism *is an $\dot{F}$-coalgebra morphism.*

An equivalent definition of an $\dot{F}$-invariant can be derived from the fibrational structure of π. For each coalgebra (I, f), there is a monotone function

$$\mathbb{E}_I \xrightarrow{\dot{F}|_I} \mathbb{E}_{FI} \xrightarrow{f^*} \mathbb{E}_I.$$

An $\dot{F}$-invariant on (I, f) is precisely a postfixed point for this function, meaning there is a morphism $X \to f^*\dot{F}|_I X$ over id_I.

Definition 3.6. *Consider the situation* (1)*. The greatest $\dot{F}$-invariant on an F-coalgebra (I, f) always exists, is called the $\dot{F}$-*coinductive invariant*, and is denoted by $\nu\dot{F}_{(I,f)}$.*

We can alternatively reach $\nu\dot{F}_{(I,f)}$ by the final sequence argument inside the fiber $\mathbb{E}_I$; this is the approach taken in [5].

In particular fibrations, coinductive invariants are often known by more specific names. For example, in **ERel**, the coinductive invariant created by the canonical relation lifting (see Definition 5.1) is *bisimilarity* (see e.g. [7]). In $\mathbf{PMet}_b$, the coinductive invariant created by a lifting is known as the *behavioural metric* on that coalgebra [5, 9].

$\dot{F}$-invariants and $\dot{F}$-invariant morphisms together form a category, in fact exactly the category $\mathbf{Coalg}(\dot{F})$. $\dot{F}$-invariants also evidently sit over F-coalgebras according to π, so we name the functor sending $\mathbf{Coalg}(\dot{F})$ to $\mathbf{Coalg}(F)$.

Definition 3.7. *In the situation* (1)*, the* underlying coalgebra functor $\mathbf{Coalg}(\pi) : \mathbf{Coalg}(\dot{F}) \to \mathbf{Coalg}(F)$ *is defined as*

$$\mathbf{Coalg}(\pi)(X, \alpha) \triangleq (\pi X, \pi\alpha), \quad \mathbf{Coalg}(\pi)(h) \triangleq \pi h.$$

4 Further Background

In this section, we recall some more advanced terms and facts from the theory of fibered categories which we will use to construct families of endoliftings for endofunctors in fibrations. We also discuss some properties of $\mathbf{CLat}_\wedge$-fibrations, particularly a canonical symmetric monoidal closed structure.

4.1 Further Details on Fibered Category Theory

A functor $\pi : \mathbb{E} \to \mathbb{B}$ is a *opfibration* if $\pi^{\mathrm{op}} : \mathbb{E}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is a fibration, and a *bifibration* if π and π^{op} are fibrations. For an opfibration π, the pullback operation of π^{op} is denoted by f_* and called *pushforward*. In a bifibration, the pullback f^* is right adjoint to the pushforward f_* [19, Lemma 9.1.2].

A convenient way to create fibrations is to form pullbacks of fibrations. This operation is called *change-of-base* of fibrations, which we describe below. Let $\pi : \mathbb{E} \to \mathbb{B}$ be a fibration, and $F : \mathbb{C} \to \mathbb{B}$ be a functor. We take the pullback of π along F in **CAT**:

$$\begin{tikzcd}
F^*\mathbb{E} \arrow[r] \arrow[d, "F^*\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "\pi"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{B}
\end{tikzcd}$$

The category $F^*\mathbb{E}$ at the top-left corner of the pullback is given as follows:

- An object is a pair (I, X) of objects $I \in \mathbb{C}$ and $X \in \mathbb{E}$ above FI.
- A morphism from (I, X) to (J, Y) is a pair $(f, \dot{f})$ of morphisms $f : I \to J$ and $\dot{f} : X \to Y$ such that $\dot{f}$ is above Ff.

The evident projection functor $F^*\pi : F^*\mathbb{E} \to \mathbb{C}$ is again a fibration and is called the *change-of-base* of π along F. If π is a partial order fibration, then so is $F^*\pi$.

A common scenario encountered in the study of fibrations is when each fiber $\mathbb{E}_I$ has a categorical structure, say $\mathcal{S}$, and pullback functors preserve these fiberwise structures. When this is the case, we say that the fibration *has fibered* $\mathcal{S}$. For instance, a fibration $\pi : \mathbb{E} \to \mathbb{B}$ has *fibered final objects* if 1) each fiber $\mathbb{E}_I$ has a final object, and 2) for any morphism $f : I \to J$, the pullback functor $f^* : \mathbb{E}_J \to \mathbb{E}_I$ preserves final objects. The fiberwise structure and the structure on the total category often have a close relationship. We state it next for the case of fibered limits.

Theorem 4.1 (Jacobs, [19])**.** *Let $\pi : \mathbb{E} \to \mathbb{B}$ be a fibration and $\mathbb{D}$ be a category. If $\mathbb{B}$ has limits of shape $\mathbb{D}$, and π has fibered limits of shape $\mathbb{D}$, then $\mathbb{E}$ also has limits of shape $\mathbb{D}$.*

The dual version of this theorem also holds, replacing the fibration with an opfibration, limits with colimits and pullbacks with pushforwards.

We also mention the preservation of fibrations by the functor-category construction:

Theorem 4.2. *For any (resp. partial order) fibration $\pi : \mathbb{E} \to \mathbb{B}$ and category $\mathbb{C}$, the postcomposition functor $\pi \circ - : [\mathbb{C}, \mathbb{E}] \to [\mathbb{C}, \mathbb{B}]$ is also a (resp. partial order) fibration.*

Proof. Consider the following situation:

$$\begin{tikzcd} \alpha^*\dot{G} \arrow[r, dotted, "\overline{\alpha}\dot{G}"] & \dot{G} & {[\mathbb{C}, \mathbb{E}]} \arrow[d, "\pi\circ-"] \\ F \arrow[r, "\alpha"'] & G & {[\mathbb{C}, \mathbb{B}]} \end{tikzcd}$$

We define the pullback $\alpha^*\dot{G}$ and the cartesian lifting $\overline{\alpha}\dot{G}$ by

$$(\alpha^*\dot{G})C \triangleq (\alpha_C)^*(\dot{G}C), \quad (\overline{\alpha}\dot{G})_C \triangleq \overline{\alpha_C}(\dot{G}C).$$

It is routine to check that this yields a cartesian lifting. $\square$

Let $\pi : \mathbb{E} \to \mathbb{C}$ and $\rho : \mathbb{F} \to \mathbb{D}$ be fibrations. A *fibration map* from π to ρ is a pair of functors $H : \mathbb{C} \to \mathbb{D}$ and $\dot{H} : \mathbb{E} \to \mathbb{F}$ such that $H\circ\pi = \rho\circ\dot{H}$, and $\dot{H}$ sends cartesian morphisms in $\mathbb{E}$ to cartesian morphisms in $\mathbb{F}$. When $\mathbb{C} = \mathbb{D}$, we also say that a functor $\dot{H} : \mathbb{E} \to \mathbb{F}$ is a fibration map from π to ρ if $(\mathrm{Id}_{\mathbb{C}}, \dot{H})$ is so. Here we relate the facts of being a right adjoint and preserving fibered meets for a fibration map between $\mathbf{CLat}_\wedge$-fibrations.

Theorem 4.3. *Let $\pi : \mathbb{E} \to \mathbf{Set}$ and $\rho : \mathbb{F} \to \mathbf{Set}$ be $\mathbf{CLat}_\wedge$-fibrations and H be a fibration map from π to ρ. Then the following are equivalent:*

1. *H preserves fibered meets, that is, for all $I \in \mathbf{Set}$, $H|_I : \mathbb{E}_I \to \mathbb{F}_I$ preserves all meets.*
2. *H has a left adjoint $G : \mathbb{F} \to \mathbb{E}$ such that $\pi\circ G = \rho$, and the unit and counit of the adjunction are above identity morphisms.*

Proof. In this proof, the (unique) $\mathbb{E}$-morphism corresponding to the inequality $X \le Y$ in the fiber $\mathbb{E}_I$ is denoted by $\lceil X \le Y \rceil$.

($1 \implies 2$) Let $X \in \mathbb{F}, Y \in \mathbb{E}$ be objects and define $I \triangleq \rho X, J \triangleq \pi Y$. Since the restriction $H|_I : \mathbb{E}_I \to \mathbb{F}_I$ of H onto the fibers above $I \in \mathbf{Set}$ preserves all meets, and since those fibers are complete lattices, the left adjoint G_I of $H|_I$ exists. We define $\eta_X \triangleq \lceil X \le HG_IX \rceil$, and show that (G_IX, η_X) is a universal arrow from X to H. Let $\dot{f} : X \to HY$ be an $\mathbb{F}$-morphism and define $f \triangleq \rho\dot{f}$. Then we obtain the factorization

$$\dot{f} = \overline{f}(HY)\circ\lceil X \le f^*(HY)\rceil = H(\overline{f}(Y))\circ\lceil X \le H(f^*Y)\rceil;$$

here we use the fact that H preserves cartesian morphisms. Next, $X \le H(f^*Y)$ implies $G_IX \le f^*Y$. Therefore we define the adjoint mate of $\dot{f}$ by $\underline{\dot{f}} \triangleq \overline{f}(Y)\circ\lceil G_IX \le f^*Y\rceil$, which is above f. Then

$$H\underline{\dot{f}}\circ\eta_X = H(\overline{f}(Y))\circ H\lceil G_IX \le f^*Y\rceil\circ\lceil X \le HG_IX\rceil = \dot{f}.$$

Let $\dot{g}$ be an $\mathbb{E}$-morphism such that $H\dot{g}\circ\eta_X = \dot{f}$ Then $\pi\dot{g} = f$ because $f = \rho\dot{f} = \rho H\dot{g}\circ\rho\eta_X = \pi\dot{g}$. From faithfulness of π, we conclude $\dot{g} = \underline{\dot{f}}$.

Thus the object part of the left adjoint of H is $X \mapsto G_{\rho X} X$. This proof shows that the bijective correspondence $\mathbb{F}(X, HY) \cong \mathbb{E}(GX, Y)$ restricts to $\mathbb{F}_f(X, HY) \cong \mathbb{E}_f(GX, Y)$ for any $f : I \to J$. This implies that the unit and counit of $G \dashv H$ is above identity. $\pi \circ G = \rho$ is easy.

(2 $\implies$ 1) Suppose that H has a left adjoint $G : \mathbb{F} \to \mathbb{E}$ such that $\pi \circ G = \rho$, and the unit η and counit ϵ of the adjunction are above identity morphisms. Then the bijective correspondence $\mathbb{E}(GX, Y) \cong \mathbb{F}(X, HY)$ can be restricted to $\mathbb{E}_f(GX, Y) \cong \mathbb{F}_f(X, HY)$ for any function $f \in \rho X \to \pi Y$ in **Set**. This particularly implies $G_I X \leq Y \iff X \leq H_I Y$ for each $I \in \mathbf{Set}$, establishing an adjunction between $\mathbb{E}_I$ and $\mathbb{F}_I$. Hence H preserves all meets.

□

4.2 Properties of $\mathbf{CLat}_\wedge$-Fibrations

We have already described the class of $\mathbf{CLat}_\wedge$-fibrations (see Section 3.4). All of the fibrations we mentioned in Example 2.1 are indeed $\mathbf{CLat}_\wedge$-fibrations, and there are many more examples.

Example 4.1. *The forgetful functors from the following categories to* **Set** *are* $\mathbf{CLat}_\wedge$*-fibrations:*

- *For a commutative unital quantale Q, Q-***Pred** *is the category of Q-valued predicates. Objects in this category are functions to Q, and morphisms from $i : I \to Q$ to $j : J \to Q$ are functions $h : I \to J$ such that $i(x) \leq j(h(x))$ holds (in Q) for any $x \in I$. This category appears in the study of up-to techniques in fibrational setting [6, Definition 7 and 8]. The forgetful functor maps i to its domain.*

- $\mathbf{PMet}_b$ *is the full subcategory of* **BVal** *consisting of b-bounded pseudometric spaces, for a fixed bound $b \in (0, \infty]$. A b-bounded pseudometric on a set I is a function $r : I \times I \to [0, b]$ which satisfies the axioms of a pseudometric:*

  1. $r(i, i) = 0$,
  2. $r(i, i') = r(i', i)$, *and*
  3. $r(i, i'') \leq r(i, i') + r(i', i'')$ *for all $i, i', i'' \in I$.*

  A pseudometric drops only the definiteness condition of a metric, so $r(i, i') = 0$ does not imply $i = i'$. This is crucial for our intended application to coalgebras where distinct states may have identical behaviours and we wish the distance between two states to reflect the difference in their behaviours only. The forgetful functor sends a pseudometric space to its underlying set.

- *For a commutative unital quantale Q regarded as a cocomplete symmetric monoidal closed category, Q-***Cat** *is the category of small Q-enriched categories and Q-enriched functors between them. The forgetful functor extracts the set of objects from small Q-enriched categories. This category is used in [4] as a generalization of metric spaces.*

Despite their simple definition, $\mathbf{CLat}_\wedge$-fibrations have many useful properties. Let $\pi : \mathbb{E} \to \mathbf{Set}$ be a $\mathbf{CLat}_\wedge$-fibration. The following properties are well-known:

- π is a split bifibration. (Each fiber is a poset and each pullback functor $f^* : \mathbb{E}_J \to \mathbb{E}_I$ has a left adjoint $f_* : \mathbb{E}_I \to \mathbb{E}_J$ by the adjoint functor theorem.)
- π is faithful and has the left adjoint $\Delta : \mathbf{Set} \to \mathbb{E}$ mapping $I \in \mathbf{Set}$ to the least element in $\mathbb{E}_I$. The unit of this adjunction is the identity morphism. Typically, ΔI corresponds to the *discrete structure* on I. For instance, ΔI in $\mathbf{Top}$ is the discrete space over I, in $\mathbf{Pre}$ it is the diagonal relation over I, and in $\mathbf{PMet}_1$ it is the discrete pseudometric over I.
- $\mathbb{E}$ has small limits and colimits that are strictly preserved by π, due to Theorem 4.1. [2]
- π uniquely lifts arbitrary limits and colimits that exist in $\mathbf{Set}$, including large ones [1, Proposition 13.15 and Proposition 21.15]. We describe this for the case of colimits. For any diagram $F : \mathbb{D} \to \mathbb{E}$ and a colimiting cocone $\{\iota_D : \pi F D \to C\}_{D \in \mathbb{D}}$ of πF in $\mathbf{Set}$, there exists a unique colimiting cocone $\{i_D : FD \to \dot{C}\}_{D \in \mathbb{D}}$ of F in $\mathbb{E}$ such that $\pi i_D = \iota_D$. The colimit $\dot{C}$ is given as $\bigvee_{D \in |\mathbb{D}|} (\iota_D)_*(FD)$. The same statement holds for coends instead of colimits.
- The change-of-base of a $\mathbf{CLat}_\wedge$-fibration $\pi : \mathbb{E} \to \mathbf{Set}$ along *any* functor $F : \mathbf{Set} \to \mathbf{Set}$ is again a $\mathbf{CLat}_\wedge$-fibration over $\mathbf{Set}$.

Another less known, but important fact is that the total category $\mathbb{E}$ of any $\mathbf{CLat}_\wedge$-fibration over $\mathbf{Set}$ carries a canonical *symmetric monoidal closed* (*SMC* for short) structure. The SMC structure on $\mathbf{Top}$ is described in [8, 30]. The following construction of the SMC structure is a reformulation of the one given in [23] using fibered category theory.

The tensor unit is $\Delta 1$.

The tensor product of $X, Y \in \mathbb{E}$ is constructed as follows. Let us define $\pi X \cdot Y$ to be the coproduct of πX-many copies of Y. We explicitly construct it above $\pi X \times \pi Y$ by

$$\pi X \cdot Y = \bigvee_{x \in \pi X} (x, -)_* Y,$$

where $(x, -) : \pi Y \to \pi X \times \pi Y$ is the function that pairs an argument with a specified $x \in \pi X$. We symmetrically define $X \cdot \pi Y$ to be the coproduct of πY-many copies of X, again constructed above $\pi X \times \pi Y$. We then define the tensor product of X and Y to be the join of these two in the fiber over $\pi X \times \pi Y$:

$$X \otimes Y = (\pi X \cdot Y) \vee (X \cdot \pi Y). \tag{2}$$

[2] In general $\mathbb{E}$ may *not* be a distributive category, that is, $X \times (-)$ may not preserve coproducts. An example $\mathbf{CLat}_\wedge$-fibration is the projection functor $\pi_2 : L \times \mathbf{Set} \to \mathbf{Set}$ where L is a non-distributive complete lattice.

This tensor product classifies bi-$\mathbb{E}$-morphisms in the following sense: for all objects $X, Y, Z \in \mathbb{E}$ and $f : \pi X \times \pi Y \to \pi Z$, the morphism f satisfies $f : X \otimes Y \dot{\to} Z$ if and only if $f(x, -) : Y \dot{\to} Z$ and $f(-, y) : X \dot{\to} Z$ holds for any $x \in \pi X$ and $y \in \pi Y$. The proof is the following:

$$\begin{aligned}
& f : X \otimes Y \dot{\to} Z \\
\iff & (f : \pi X \cdot Y \dot{\to} Z) \text{ and } (f : X \cdot \pi Y \dot{\to} Z) \\
\iff & (\forall x \in \pi X \ . \ f(x, -) : Y \dot{\to} Z) \text{ and } (\forall y \in \pi Y \ . \ f(-, y) : X \dot{\to} Z).
\end{aligned}$$

The closed structure of $X, Y \in \mathbb{E}$ is constructed as follows. We first construct the product $\pi X \pitchfork Y$ of πX-many copies of Y above $\mathbf{Set}(\pi X, \pi Y)$ by

$$\pi X \pitchfork Y = \bigwedge_{x \in \pi X} (-(x))^* Y,$$

where $-(x) : \mathbf{Set}(\pi X, \pi Y) \to \pi Y$ is the function that evaluates an argument function with a specified $x \in \pi X$. We then define the closed structure $X \multimap Y$ to be the pullback of $\pi X \pitchfork Y$ along the morphism mapping $\pi_{X,Y} : \mathbb{E}(X, Y) \to \mathbf{Set}(\pi X, \pi Y)$ of π:

$$X \multimap Y = \pi^*_{X,Y}(\pi X \pitchfork Y). \tag{3}$$

Proposition 4.1. *The functor $\pi : \mathbb{E} \to \mathbf{Set}$ and its left adjoint $\Delta : \mathbf{Set} \to \mathbb{E}$ are strict symmetric monoidal (for $\mathbf{Set}$ we take the cartesian monoidal structure).*

Proof. That π being strict symmetric monoidal is immediate. To show that Δ is so, we first check $I \cdot \Delta J = \Delta(I \times J)$ for any $I, J \in \mathbf{Set}$:

$$\mathbb{E}(I \cdot \Delta J, -) \cong \mathbf{Set}(I, \mathbb{E}(\Delta J, -)) \cong \mathbf{Set}(I, \mathbf{Set}(J, \pi -)) \cong \mathbf{Set}(I \times J, \pi -) \cong \mathbb{E}(\Delta(I \times J), -).$$

We thus conclude $I \cdot \Delta J = \Delta(I \times J)$. By a symmetrical argument, $\Delta I \cdot J = \Delta(I \times J)$ also holds. Therefore $\Delta I \otimes \Delta J = (I \cdot \Delta J) \vee (\Delta I \cdot J) = \Delta(I \times J)$. That Δ maps $1 \in \mathbf{Set}$ to the tensor unit is trivial. □

Example 4.2. *We illustrate the bifibrational structure of* $\mathbf{BVal}$*. Let us first recall the order relation in its fibers. For any $I \in \mathbf{Set}$ and $(I, r), (I, s) \in \mathbf{BVal}_I$, the following are equivalent:*

1. *$(I, r) \le (I, s)$ holds in $\mathbf{BVal}_I$,*
2. *id_I is a nonexpansive function from (I, r) to (I, s), and*
3. *$s(x, y) \le r(x, y)$ holds for all $x, y \in I$.*

Note the apparent disparity between 1 and 3: though $(I, r) \leq (I, s)$ in the fiber order, s has smaller values than r pointwise. As a consequence, the meet in the fiber is computed by the pointwise numerical sup: $\bigwedge_{i\in\lambda}(I, r_i) = (I, r')$ where $r'(x, y) = \sup_{i\in\lambda} r_i(x, y)$.

Next, let $(I, r) \in \mathbf{BVal}$ and $H \xrightarrow{f} I \xrightarrow{g} J$ be functions. The pullback $g^*(I, r) \triangleq (H, q)$ and the pushforward $f_*(I, s) \triangleq (J, s)$ are given by

$$q(x, y) = r(f(x), f(y)), \quad s(x, y) = \inf_{\substack{g(p)=x \\ g(q)=y}} r(p, q).$$

The fibrational construction of the canonical SMC structure on $\mathbf{BVal}$ yields the following tensor product and closed structure:

$$(I, r) \otimes (J, s) = \left(I \times J, \quad \lambda((x, y), (x', y')) \,.\, \begin{cases} \infty & x \neq x' \text{ and } y \neq y' \\ s(y, y') & x = x' \text{ and } y \neq y' \\ r(x, x') & x \neq x' \text{ and } y = y' \\ \min(r(x, x'), s(y, y')) & x = x' \text{ and } y = y' \end{cases} \right)$$

$$(I, r) \multimap (J, s) = \left(\mathbf{BVal}((I, r), (J, s)), \quad \lambda(f, f') \,.\, \sup_{x\in I} s(\pi f(x), \pi f'(x)) \right)$$

Example 4.3. *We look at the fibrational structure of Q-$\mathbf{Cat}$ for a commutative unital quantale $(Q, \leq, 1, \cdot)$. In this example, π is a shorthand of the functor $\pi_{Q\text{-}\mathbf{Cat}} : Q\text{-}\mathbf{Cat} \to \mathbf{Set}$ extracting the set of objects of small Q-enriched categories (Example 4.1). The pullback of $X \in Q$-$\mathbf{Cat}$ along a function $f : I \to \pi X$ is defined to be the Q-enriched category f^*X whose set of objects is I, and whose hom-objects are defined by $f^*X(i, i') = X(fi, fi')$. The fibered meets of $X_i \in Q$-$\mathbf{Cat}_I$ for a set I is the Q-enriched category $\bigwedge X_i$ whose set of objects is I and whose hom-objects are given by $(\bigwedge X_i)(x, y) = \bigwedge X_i(x, y)$. From these data, $\pi : Q$-$\mathbf{Cat} \to \mathbf{Set}$ is a $\mathbf{CLat}_\wedge$-fibration.*

We next derive the SMC structure on Q-$\mathbf{Cat}$ using (2) and (3). We first compute the closed structure. Following the equation (3), the set of objects of $X \multimap Y$ is Q-$\mathbf{Cat}(X, Y)$, and its hom-objects are given by

$$(X \multimap Y)(f, g) = \bigwedge_{x\in\pi X} Y(f(x), g(x)).$$

This is the same as the Q-enriched category $[X, Y]$ of Q-enriched functors given in [4, Section 2.3 Equation (2.8)]. We next define the tensor product $X \otimes Y$ to be the Q-enriched category whose set of objects is $\pi X \times \pi Y$, and whose hom-objects are given by

$$X \otimes Y((x, y), (x', y')) = X(x, x') \cdot Y(y, y').$$

This tensor product also appears in [4, Section 2.3]. We can easily check that this Q-enriched *category classifies bi-Q-enriched functors, hence coincides with the one given as* (2)*. To summarize, the SMC structure on Q-*$\mathbf{Cat}$ *derived from the* $\mathbf{CLat}_\wedge$*-fibration* $\pi_{Q\text{-}\mathbf{Cat}} : Q\text{-}\mathbf{Cat} \to \mathbf{Set}$ *coincides with the one presented in [4, Section 2.3].*

4.3 Liftings in $\mathbf{CLat}_\wedge$-Fibrations

Here we again consider the situation (1), depicted as

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\dot{F}"] \arrow[d, "\pi"'] & \mathbb{E} \arrow[d, "\pi"] \\ \mathbf{Set} \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

$\dot{F}$-coinductive invariants give final objects within each fiber category, but there is no assurance a final object exists in the total category, nor that final objects are preserved by coalgebra morphisms. The next result, which reorganizes results presented in [15, Section 4], sets out some conditions when these hold.

Theorem 4.4. *Consider the situation* (1)*. If* $(F, \dot{F})$ *is a fibration map,*

1. *[15, Proposition 4.1] The underlying coalgebra functor* $\mathbf{Coalg}(\pi) : \mathbf{Coalg}(\dot{F}) \to \mathbf{Coalg}(F)$ *is a fibration where pullbacks are the same as in the fibration π.*
2. *Each pullback functor of* $\mathbf{Coalg}(\pi)$ *preserves final objects (hence* $\mathbf{Coalg}(\dot{F})$ *has fibered final objects).*
3. *If additionally* $\mathbf{Coalg}(F)$ *has a final object νF, then* $\mathbf{Coalg}(\dot{F})$ *has a final object.*

Proof. 1. Let (I, f) and (J, g) be F-coalgebras, $\varphi : (I, f) \to (J, g)$ be an F-coalgebra morphism, and $(Y, \dot{g})$ be an $\dot{F}$-coalgebra above (J, g). Since π is a fibration, the function $\varphi : I \to J$ and the $\mathbb{E}$-object Y yield a π-cartesian morphism $\overline{\varphi}(Y) : \varphi^* Y \to Y$ above $\varphi : I \to J$. We will find an $\dot{F}$-coalgebra structure on $\varphi^* Y$ making $\overline{\varphi}(Y)$ a $\mathbf{Coalg}(\pi)$-cartesian morphism.

Since $\dot{F}$ preserves cartesian morphisms, $\dot{F}(\overline{\varphi}(Y)) : \dot{F}(\varphi^* Y) \to \dot{F} Y$ is cartesian above $F\varphi : FI \to FJ$. Since $F\varphi \circ f = g \circ \varphi$, from the universal property of $F\varphi$ we obtain a mediating morphism $m : \varphi^* Y \to \dot{F}(\varphi^* Y)$ above f, which is the pullback coalgebra of $(Y, \dot{g})$ along φ.

2. Let $h : (I, f) \to (J, g)$ be an F-coalgebra morphism. Then for each $X \in \mathbb{E}_J$ we have the following equalities between objects:

$$h^*(g^*(\dot{F} X)) = (g \circ h)^* \dot{F} X = (F h \circ f)^* \dot{F} X = f^*((F h)^*(\dot{F} X)) = f^*(\dot{F}(h^* X)).$$

This, together with h^* preserving all meets, implies that h^* maps the final sequence of $f^* \circ \dot{F}$ to the final sequence of $g^* \circ \dot{F}$. Therefore $h^*(\nu(f^* \circ \dot{F})) = \nu(g^* \circ \dot{F})$.

3. Immediate from Theorem 4.1. It is given as the final object $\nu \dot{F}_{\nu F}$ above the final coalgebra. $\square$

For items 2 and 3 of this theorem, see also [15, Corollary 4.3]. This is a fibered counterpart of some results in Section 6 of [5]. To see this, we instantiate Theorem 4.4 with π being the forgetful functor from $\mathbf{PMet}_b$, and F being an endofunctor on $\mathbf{Set}$ having a final F-coalgebra νF. If $(F, \dot{F})$ is a fibration map (that is, $\dot{F}$ preserves isometries), then

- Theorem 6.1 in [5] is equivalent to the conclusion of (this instance of) item 3 of Theorem 4.4.
- Let (I, f) be an F-coalgebra, and $!_I : I \to \nu F$ be the unique F-coalgebra morphism. The *behavioural distance* of I in [5] corresponds to the pullback $!_I^*(\nu \dot{F}_{\nu F})$ in our fibrational language.
- Theorem 6.2 in [5] corresponds to $\nu F_I = !_I^*(\nu \dot{F}_{\nu F})$, which follows from (this instance of) item 2 of Theorem 4.4.

5 Constructions of Endoliftings along $\mathbf{CLat}_\wedge$-Fibrations

There are many examples of endoliftings of endofunctors in well-known fibrations, such as the fibration of relations or pseudometrics. Some of these endoliftings even form classes which cover all functors, such as the canonical relation endolifting [7, 17] or the generalized Kantorovich liftings of [5], which ensure *every* functor has an endolifting in **ERel** and $\mathbf{PMet}_b$ respectively. In this section, we generalize various constructions known in particular fibrations to arbitrary $\mathbf{CLat}_\wedge$-fibrations. Throughout this section we let $\pi : \mathbb{E} \to \mathbf{Set}$ be a $\mathbf{CLat}_\wedge$-fibration. A summary of the subsequent sections are in order:

Section 5.1 We review the construction of endoliftings of polynomial functors along fibrations, first studied by Hermida and Jacobs [17]. These endoliftings are often used for defining bisimulations for deterministic transition systems, such as deterministic finite automata. The main ingredients of this construction are products and coproducts in $\mathbb{E}$ that are strictly preserved by $\pi : \mathbb{E} \to \mathbf{Set}$.

Section 5.2 and 5.3 We introduce the pullback and pushforward constructions of endoliftings along natural transformations; they are not mentioned very often in the literature[3]. As demonstrated in the subsequent sections, these constructions yield nontrivial endoliftings. We illustrate this by constructing the *Hausdorff pseudometric* as the pushforward of the

[3]Pullbacks of monad liftings are used in [11, 20, 21].

lifting of the list polynomial functor along the quotient natural transformation into the finite powerset functor.

Section 5.4 In [6], Bonchi et al. introduce *Wasserstein lifting* as a generalization of the Wasserstein metric on probability measures; this is a part of their study on *fibrational quantitative up-to techniques*. We reformulate their Wasserstein lifting in the general $\mathbf{CLat}_\wedge$-fibration π using Hermida's *adjunction lifting* [16] together with pushforward of liftings.

Section 5.5 In [21], a lifting method of monads called *codensity lifting* is introduced. In this paper we give an endofunctor version of the codensity lifting along the $\mathbf{CLat}_\wedge$-fibration π. It includes Baldan et al.'s *Kantorovich lifting* [5], which further includes *Kantorovich metric*, as an instance. A recent work by Komorida et al. [25] points out that the bisimulation relation using codensity lifting has a game-theoretic characterization.

Section 5.6 In [4], Balan et al. gives a construction of endoliftings to the category Q-$\mathbf{Cat}$ of quantale-enriched small categories through *enriched left Kan extension*. At first sight, their construction looks specific to the SMC structure of Q-$\mathbf{Cat}$. In fact, as we pointed out in Example 4.3, the SMC structure of Q-$\mathbf{Cat}$ is an instance of the SMC structure on the total category $\mathbb{E}$ of the $\mathbf{CLat}_\wedge$-fibration π. Leveraging this fact, we give a calculation of the enriched left Kan extension using the colimit-lifting property of π.

5.1 Lifting Polynomial Functors

The fundamental class of endofunctors is *polynomial functors*. We discuss endoliftings of polynomial endofunctors along the $\mathbf{CLat}_\wedge$-fibration π. The material of this section is an adaptation of the lifting techniques studied by Hermida and Jacobs [17, Section 2] in $\mathbf{CLat}_\wedge$-fibrations. For the systematic treatment of polynomial functors in general categories, we first introduce a syntactic structure called *polynomials* in a category $\mathbb{C}$. They form the class $\mathbf{P}(\mathbb{C})$, and are defined by the following BNF:

$$\mathbf{P}(\mathbb{C}) \ni P ::= C \mid \mathrm{Id} \mid \prod_{i \in I} P_i \mid \coprod_{i \in I} P_i \quad (C \in \mathbb{C}, I \in \mathbf{Set})$$

These are merely syntactic expressions and will be interpreted as endofunctors on $\mathbb{C}$. The product and coproduct of two polynomials are denoted by the infix operators $\times, +$ respectively. We say that a polynomial P is *finitary* if indexing sets of $\prod$ and $\coprod$ in P are all finite and countable, respectively. Examples of polynomials include:

- Let $A \in \mathbf{Set}$. Then $A \times \mathrm{Id}$ is a finitary polynomial in $\mathbf{Set}$.
- Let $(I, r) \in \mathbf{BVal}$ and $A \in \mathbf{Set}$. Then $(I, r) \times \prod_{a \in A} \mathrm{Id}$ is a polynomial in $\mathbf{BVal}$.
- $P_{\mathrm{list}} \triangleq \coprod_{i \in \mathbb{N}} \left(\prod_{j \in \{0,\dots,i-1\}} \mathrm{Id}\right)$ is a finitary polynomial in any category $\mathbb{C}$.

We next extend a functor $F : \mathbb{C} \to \mathbb{D}$ to the function $\mathbf{P}(F) : \mathbf{P}(\mathbb{C}) \to \mathbf{P}(\mathbb{D})$ between classes of polynomials. It replaces each $\mathbb{C}$-object C in a given polynomial in $\mathbb{C}$ with FC. It is easy to see that $\mathbf{P}(\mathrm{Id}_{\mathbb{C}}) = \mathrm{id}_{\mathbf{P}(\mathbb{C})}$ and $\mathbf{P}(G) \circ \mathbf{P}(F) = \mathbf{P}(G \circ F)$.

Let $\mathbb{C}$ be a category with small products and small coproducts. For a polynomial $P \in \mathbf{P}(\mathbb{C})$, we define its *interpretation* to be an endofunctor on $\mathbb{C}$ with the obvious recursion. The interpretation of a polynomial P in $\mathbb{C}$ is denoted $\mathbb{C}[\![P]\!]$. An endofunctor on $\mathbb{C}$ is *(finitary) polynomial* if it is an interpretation of a (finitary) polynomial in $\mathbb{C}$. For instance, the polynomial P_{list} determines the list functor L on $\mathbf{Set}$:

$$LI \triangleq \mathbf{Set}[\![P_{\mathrm{list}}]\!](I) = 1 + I + I^2 + \cdots .$$

This syntactic structure allows us to formalize two functors having essentially the same shape.

Proposition 5.1. *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with small products and small coproducts, and $F : \mathbb{C} \to \mathbb{D}$ be a functor strictly preserving small products and small coproducts. Then we have $F \circ \mathbb{C}[\![P]\!] = \mathbb{D}[\![\mathbf{P}(F)(P)]\!] \circ F$.*

Proof. Easy induction. □

Corollary 5.1. *For any polynomial P in $\mathbb{E}$, $\mathbb{E}[\![P]\!]$ is an endolifting of $\mathbf{Set}[\![\mathbf{P}(\pi)(P)]\!]$ along π. Especially, for any polynomial P in $\mathbf{Set}$, $\mathbb{E}[\![\mathbf{P}(\Delta)(P)]\!]$ is an endolifting of $\mathbf{Set}[\![P]\!]$.*

Proof. Recall that $\mathbf{CLat}_\wedge$-fibrations have small products and coproducts that are strictly preserved by π (Section 4.2). Moreover, for any $P \in \mathbf{P}(\mathbf{Set})$, we have $P = \mathbf{P}(\pi \circ \Delta)(P) = \mathbf{P}(\pi)(\mathbf{P}(\Delta)(P))$. □

The canonical relation liftings of Hermida and Jacobs for polynomial endofunctors restrict the constants available in the polynomials.

Definition 5.1. *The* canonical relation liftings *are the interpretations of polynomials in* **ERel** *where the constants are all the diagonal relation: (I, Δ_I) for some set I. In more detail, for a polynomial P in* **Set***, we define $\mathbf{Rel}(P) = \mathbf{P}(\imath \circ \Delta_{\mathbf{Pre}})(P)$ where $\imath : \mathbf{Pre} \to \mathbf{ERel}$ is the inclusion of preorders in endorelations. The canonical relation lifting of $\mathbf{Set}[\![P]\!]$ is the interpretation of the polynomial $\mathbf{Rel}(P)$ in* **ERel***.*

Note that the canonical relation lifting is also commonly defined using an epi-mono factorization in $\mathbf{Set}$. For the case of polynomial endofunctors, these definitions coincide, so we use this definition for convenience.

Example 5.1. *We regard P_{list} as a polynomial in* **ERel***, and define $L^{\mathbf{ERel}}$ to be the interpretation of P_{list} in* **ERel***. Then $L^{\mathbf{ERel}}$ is an endolifting of L along $\pi_{\mathbf{ERel}}$ by Corollary 5.1. For $(I, R) \in$* **ERel***, the relation part of $L^{\mathbf{ERel}}(I, R)$ relates two lists $(k_1, \cdots, k_m), (\ell_1, \cdots, \ell_n) \in LI$ of arbitrary length if and only if $m = n$ (they come from the same index in the coproduct), and $(k_i, \ell_i) \in R$ holds for $0 \leq i < \mathrm{len}(k)$.*

5.2 Pullback and Pushforward of Endoliftings

We consider extending the pullback and pushforward mechanisms of π to endoliftings. Let $\dot{F}$ be an endolifting of an endofunctor F on **Set** along π, and $H \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be natural transformations. We introduce the following pointwise pushforward and pullback in the fibration $\pi : \mathbb{E} \to \mathbf{Set}$:

$$(\alpha^* \dot{F})X \triangleq (\alpha_{\pi X})^*(\dot{F}X) \qquad\qquad (\beta_* \dot{F})X \triangleq (\beta_{\pi X})_*(\dot{F}X).$$

They are pullback and pushforward in the following fibration $l_\pi : \mathbf{Lift}(\pi) \to [\mathbf{Set}, \mathbf{Set}]$. It is the change-of-base of the fibration $\pi \circ - : [\mathbb{E}, \mathbb{E}] \to [\mathbb{E}, \mathbf{Set}]$ (see Theorem 4.2) along $- \circ \pi : [\mathbf{Set}, \mathbf{Set}] \to [\mathbb{E}, \mathbf{Set}]$:

$$\begin{tikzcd}
\mathbf{Lift}(\pi) \arrow[r] \arrow[d, "l_\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbb{E}, \mathbb{E}]} \arrow[d, "\pi \circ -"] \\
{[\mathbf{Set}, \mathbf{Set}]} \arrow[r, "- \circ \pi"'] & {[\mathbb{E}, \mathbf{Set}]}
\end{tikzcd} \tag{4}$$

The category $\mathbf{Lift}(\pi)$ at the top-left corner is described as follows: objects are pairs $(F, \dot{F})$ of an endofunctor F on **Set** and one of its endoliftings $\dot{F} : \mathbb{E} \to \mathbb{E}$ along π. The object $(F, \dot{F}) \in \mathbf{Lift}(\pi)$ may simply be denoted by $\dot{F}$ if F is obvious from the context. A morphism from $(F, \dot{F})$ to $(G, \dot{G})$ is a pair $(\alpha, \dot{\alpha})$ of natural transformations $\alpha : F \to G$ and $\dot{\alpha} : \dot{F} \to \dot{G}$ such that $\pi \circ \dot{\alpha} = \alpha \circ \pi$. The fibration l_π is a partial order bifibration (see Section 4.1). The order relation of each fiber is given by: $(F, \dot{F}) \leq (F, \ddot{F})$ if and only if for all $X \in \mathbb{E}$, $\dot{F}X \leq \ddot{F}X$ holds in $\mathbb{E}_{F\pi X}$. We adopt this as the ordering relation on endoliftings of F. The pullback and pushforward of $(F, \dot{F}) \in \mathbf{Lift}(\pi)$ along $H \xrightarrow{\alpha} F \xrightarrow{\beta} G$ are given by $\alpha^*(F, \dot{F}) = (H, \alpha^* \dot{F})$ and $\beta_*(F, \dot{F}) = (G, \beta_* \dot{F})$ respectively.

Definition 5.2. *A functor F : **Set** $\to$ **Set** is said to be* finitary *if there is a finitary polynomial functor P : **Set** $\to$ **Set** and an epic*[4] *natural transformation $\alpha : P \Rightarrow F$. In this case, F is called the* quotient *of P by α.*

Note that a more common definition of "finitary functor" is that the functor preserves filtered colimits. That the definition above coincides with this more usual one was shown to hold in locally finitely presentable categories [2].

The advantage of this definition is it makes immediately clear that every finitary endofunctor on **Set** has an endolifting in every $\mathbf{CLat}_\wedge$-fibration: polynomial functors have endoliftings and these endoliftings can be pushed forward along the quotienting natural transformations.

Example 5.2. *(Continued from the previous Example) The finite powerset functor P_{fin} is the quotient of the list functor L. The quotient map is the natural transformation* $\mathrm{set}_I : LI \to P_{\mathrm{fin}} I$

[4] Meaning each component $\alpha_I : PI \to FI$ is an epi in **Set**, or that α is an epi in the functor category [**Set**, **Set**]. MacLane [26] shows these two definitions of "epic natural transformation" are equivalent for transformations between functors into **Set**.

given by $\mathrm{set}_I(i_1, \dots, i_n) = \{i_1, \dots, i_n\}$*. The pushforward of* $L^{\mathbf{ERel}}$ *along* set *yields the endolifting of* P_{fin} *that creates bisimilarity as its coinductive invariant. Explicitly, the pushforward* $P_{\mathrm{fin}}^{\mathbf{ERel}} \triangleq \mathrm{set}_*(L^{\mathbf{ERel}})$ *acts on objects as*

$$P_{\mathrm{fin}}^{\mathbf{ERel}}(I, R) = (P_{\mathrm{fin}} I, \{(J, K) \mid (\forall j \in J \,.\, \exists k \in K \,.\, (j, k) \in R) \text{ and } (\forall k \in K \,.\, \exists j \in J \,.\, (j, k) \in R)\}).$$

5.3 The Hausdorff Pseudometric

We next demonstrate a more elaborate example of the *Hausdorff pseudometric* as an endolifting of the finite powerset functor P_{fin} along $\pi_{\mathbf{BVal}} : \mathbf{BVal} \to \mathbf{Set}$.

First, we regard P_{list} as a polynomial in **BVal**, and let $L^{\mathbf{BVal}}$ be its interpretation in **BVal**. This is an endolifting of the list functor L along $\pi_{\mathbf{BVal}}$, and its object part satisfies the following:

$$L^{\mathbf{BVal}}(I, d) = (LI, d^*) \qquad \text{where } d^*(k, h) = \begin{cases} \max\limits_{0 \le i < \mathrm{len}(k)} d(k_i, h_i) & \text{if } \mathrm{len}(k) = \mathrm{len}(h) \\ \infty & \text{if } \mathrm{len}(k) \neq \mathrm{len}(h) \end{cases}$$

We then take the pushforward $P_{\mathrm{fin}}^{\mathbf{BVal}} \triangleq \mathrm{set}_*(L^{\mathbf{BVal}})$ as we have done in Example 5.2. In Example 4.2, we found pushforward in **BVal** explicitly by $P_{\mathrm{fin}}^{\mathbf{BVal}}(I, d) = (P_{\mathrm{fin}} I, \mathcal{H}' d)$ where

$$\mathcal{H}' d(K, H) = \inf_{\substack{k \in LI :\ \mathrm{set}(k) = K \\ h \in LI :\ \mathrm{set}(h) = H}} d^*(k, h) \tag{5}$$

We have denoted this distance $\mathcal{H}' d$ since it turns out to be equal to the usual Hausdorff distance. However, this is not the usual formulation for the Hausdorff distance, so we spend the remainder of this section describing why our formulation is equivalent to the original.

The usual definition of Hausdorff distance for a metric space is

$$\mathcal{H} d(K, H) = \max\left(\sup_{y \in K} \inf_{z \in H} d(y, z), \sup_{z \in H} \inf_{y \in K} d(y, z) \right) \tag{6}$$

where $K, H \subseteq I$. Typically the Hausdorff distance is also restricted to nonempty compact subsets of the metric space so that $\mathcal{H} d$ is truly a metric. Since we are not interested in obtaining a metric, we do not make this requirement on the domain of $\mathcal{H} d$, but we do require that K and H are finite since we want a **BVal** object over $P_{\mathrm{fin}} I$. The finiteness of K and H allows us to change sup and inf above to max and min, respectively.

Notation for lists. We will be working with lists fairly intensively in this section, so we need to set up some notation. List concatenation will be denoted by $h_1 + h_2$. We assume that finite sets have been given an arbitrary but fixed ordering. More precisely, we fix a section of set_I which we call $\mathrm{list}_I : P_{\mathrm{fin}} I \to LI$. An expression like $(f(k))_{k \in K}$ where K is a finite set and

$f : K \to X$ denotes a list from LX, namely $f^*(\mathrm{list}_I(K))$. A particular example is $(k)_{k\in K}$, which is alternative notation for $\mathrm{list}_I(K)$.

Lemma 5.1. *Suppose $K, H \in P_{\mathrm{fin}} I$. Then $d^*\left((k)_{k\in K}, (\arg\min_{h\in H} d(k,h))_{k\in K}\right) = \sup_{y\in K}\inf_{z\in H} d(y,z)$, and $d^*\left((\arg\min_{k\in K} d(k,h))_{h\in H}, (h)_{h\in H}\right) = \sup_{z\in H}\inf_{y\in K} d(y,z)$.*

Proof.

$$\begin{aligned} d^*\left((k)_{k\in K}, (\arg\min_{h\in H} d(k,h))_{k\in K}\right) &= \max_{k\in K} d(k, \arg\min_{h\in H} d(k,h)) \\ &= \max_{k\in K}\min_{h\in H} d(k,h) \end{aligned}$$

And similarly for the other claim. □

Lemma 5.2. *Suppose $h_1, h_2, k_1, k_1 \in LI$ satisfy $\mathrm{len}(h_1) = \mathrm{len}(k_1)$ and $\mathrm{len}(h_2) = \mathrm{len}(k_2)$. Then $d^*(h_1 + h_2, k_1 + k_2) = \max(d^*(h_1,k_1), d^*(h_2,k_2))$.*

Given finite sets $K, H \subseteq I$, we define $s(K,H) \in LK$ to be $(k)_{k\in K} + (\arg\min_{k\in K} d(k,h))_{h\in H}$, and we define $t(K,H) \in LH$ to be $(\arg\min_{h\in H} d(k,h))_{k\in K} + (h)_{h\in H}$. Combining the previous two lemmas with this definition, we obtain a characterization of the Hausdorff distance.

Proposition 5.2. *$d^*(s(K,H), t(K,H)) = \mathcal{H}d(K,H)$ for all $K, H \in P_{\mathrm{fin}} I$.*

Next, we claim that these special lists $s(K,H)$ and $t(K,H)$ realize the minimum in $\mathcal{H}'d(K,H)$.

Proposition 5.3. *$\mathcal{H}'d(K,H) = d^*(s(K,H), t(K,H))$ for all $K, H \in P_{\mathrm{fin}} I$.*

Proof. Since $\mathrm{set}_I(s(K,H)) = K$ and $\mathrm{set}_I(t(K,H)) = H$, the distance $d^*(s(K,H), t(K,H))$ is greater than or equal to the infimum on the left. Therefore, it suffices to show that $d^*(s(K,H), t(K,H))$ is a lower bound for $\{d^*(k,h) : \mathrm{set}_I(k) = K \text{ and } \mathrm{set}_I(h) = H\}$.

Suppose for contradiction there are lists $k, h \in LI$ such that $\mathrm{set}_I(k) = K$, $\mathrm{set}_I(h) = H$ and $d^*(k,h) < d^*(s(K,H), t(K,H))$. This inequality immediately implies $\mathrm{len}(k) = \mathrm{len}(h)$.

Let i be the index such that $d^*(s(K,H), t(K,H)) = d(s(K,H)_i, t(K,H)_i)$. To be members of these lists, there are two possibilities: (A) $s(K,H)_i = \arg\min_{k\in K} d(k, t(K,H)_i)$ or (B) $t(K,H)_i = \arg\min_{h\in H} d(s(K,H)_i, h)$.

Suppose (A). Then the fact that $\mathrm{set}_I(h) = H$ means there is an index j with $0 \le j < \mathrm{len}(h)$ such that $h_j = t(K,H)_i$. Now we are ready to find our contradiction:

$$\begin{aligned} d(k_j, h_j) &\le d^*(k,h) < d^*(s(K,H), t(K,H)) = d(s(K,H)_i, t(K,H)_i) \\ &= d(\arg\min_{y\in K} d(y, t(K,H)_i), t(K,H)_i) = \min_{y\in K} d(y, t(K,H)_i) = \min_{y\in K} d(y, h_j). \end{aligned}$$

The argument in case (B) is similar, using $\mathrm{set}_I(k) = K$ instead. □

Combining the previous two propositions, we obtain the following theorem.

Theorem 5.1. *Suppose K and H are finite subsets of a set I, and I is equipped with a binary valuation $d : I \times I \to \mathbb{R}^+$. Then*

$$\mathcal{H}d(K,H) \overset{(6)}{=} \max\left(\sup_{y\in K}\inf_{z\in H} d(y,z), \sup_{z\in H}\inf_{y\in K} d(y,z)\right) = \inf_{\substack{k\in LI:\ \mathrm{set}(k)=K\\ h\in LI:\ \mathrm{set}(h)=H}} d^*(k,h) \overset{(5)}{=} \mathcal{H}'d(K,H).$$

This theorem can also be obtained using the game theoretic interpretation of the two versions of the Hausdorff distance and the equivalence of two related games. The interested reader can find a sketch of this alternative proof in the conference version [28].

5.4 Wasserstein Lifting for $\mathbf{CLat}_\wedge$-Fibrations

In [6], Bonchi et al. introduced a technique to transfer predicate liftings of endofunctors to their relational liftings (which they call *Wasserstein lifting*). In this section, we reformulate their transfer technique in general $\mathbf{CLat}_\wedge$-fibration using Hermida's adjunction lifting result.

We first sketch their transfer technique. Let Q be a commutative unital quantale, and consider the forgetful functor $\pi_{Q\text{-}\mathbf{Rel}} : Q\text{-}\mathbf{Rel} \to \mathbf{Set}$ (Example 4.1), which is a $\mathbf{CLat}_\wedge$-fibration. We derive the category of Q-valued endorelations by the following change-of-base (Section 4.1) and name the derived fibration $\pi_{Q\text{-}\mathbf{Rel}}$:

$$\begin{tikzcd}
Q\text{-}\mathbf{Rel} \arrow[r] \arrow[d, "\pi_{Q\text{-}\mathbf{Rel}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q\text{-}\mathbf{Pred} \arrow[d, "\pi_{Q\text{-}\mathbf{Pred}}"] \\
\mathbf{Set} \arrow[r, "(-)^2"'] & \mathbf{Set}
\end{tikzcd}$$

Let F be an endofunctor on $\mathbf{Set}$ and $\dot{F}$ be an endolifting of F along $\pi_{Q\text{-}\mathbf{Pred}}$. The *Wasserstein lifting* $\overline{F}$ corresponding to $\dot{F}$ is defined to be the following endolifting of F along $\pi_{Q\text{-}\mathbf{Rel}}$ [6, Section 5.2]:

$$\overline{F}(I,X) \triangleq (FI, \langle F\pi_1, F\pi_2\rangle_*(\dot{F}X)); \tag{7}$$

the right hand side is indeed an object in Q-$\mathbf{Rel}$ because the pushforward is above $(FI)^2$:

$$\begin{tikzcd}
\dot{F}X \arrow[r, dotted] & \langle F\pi_1, F\pi_2\rangle_*(\dot{F}X) & Q\text{-}\mathbf{Pred} \arrow[d, "\pi_{Q\text{-}\mathbf{Pred}}"] \\
F(I^2) \arrow[r, "\langle F\pi_1, F\pi_2\rangle"'] & (FI)^2 & \mathbf{Set}
\end{tikzcd}$$

We extend the Wasserstein lifting to general $\mathbf{CLat}_\wedge$-fibrations. For this, we give a new formulation of the Wasserstein lifting using Hermida's adjoint lifting result. We employ the adjunction $(L \dashv R : \mathbf{Set} \to \mathbf{Set}, \eta, \epsilon)$ given by $RI = I^2$ and $LI = 2 \times I$. Remark that any endofunctor F on $\mathbf{Set}$ comes with a natural transformation $\theta : L \circ F \circ R \to F$ given by the adjoint mate of $\langle F\pi_1, F\pi_2\rangle : F\circ R \to R\circ F$.

Starting from a $\mathbf{CLat}_\wedge$-fibration $\pi : \mathbb{E} \to \mathbf{Set}$, we derive the fibration of endorelations by the change-of-base of π along R:

$$\begin{tikzcd}
\mathbf{ERel}(\mathbb{E}) \arrow[r, "\dot{R}"] \arrow[d, "r_\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "\pi"] \\
\mathbf{Set} \arrow[r, shift left, "R"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, shift left, "L"]
\end{tikzcd}$$

The total category of the resulting fibration $r_\pi : \mathbf{ERel}(\mathbb{E}) \to \mathbf{Set}$ is described as follows:

- An object is a pair (I, X) where $X \in \mathbb{E}_{I^2}$.
- A morphism from (I, X) to (J, Y) is a function $f : I \to J$ such that $f^2 : X \dot{\to} Y$ holds (in the fibration π).

This category is an analogy of Q-**Rel** in [6, Section 4].

The leg $\dot{R} : \mathbf{ERel}(\mathbb{E}) \to \mathbb{E}$ of the change-of-base has a left adjoint $\dot{L} : \mathbb{E} \to \mathbf{ERel}(\mathbb{E})$. Its object part is given by $\dot{L}X = (LI, (\eta_I)_* X)$, where η is the unit of the adjunction $L \dashv R$. This left adjoint is derived from Hermida's result on lifting of left adjoints by opfibrations; see [16, Corollary 3.2.5] for details. The adjunction $\dot{L} \dashv \dot{R}$ is a *lift* of $L \dashv R$, that is, $\pi\circ\dot{R} = R\circ r_\pi$ and $r_\pi\circ\dot{L} = L\circ\pi$ hold.

Let F be an endofunctor on **Set** and $\dot{F}$ be an endolifting of F along π. Then $\dot{L}\circ\dot{F}\circ\dot{R}$ is an endolifting of $L\circ F\circ R$ along r_π. We define $W(\dot{F})$, which is now a lifting of F along r_π, to be the pushforward of $\dot{L}\circ\dot{F}\circ\dot{R}$ along $\theta : L\circ F\circ R \to F$.

Theorem 5.2. *For any endofunctor F on* **Set** *and one of its endoliftings $\dot{F}$ along π, we have*

$$W(\dot{F})(I, X) = (FI, \langle F\pi_1, F\pi_2\rangle_*(\dot{F}X)).$$

Proof. Let $(I, X) \in \mathbf{ERel}(\mathbb{E})$. By unfolding the definition, we obtain

$$W(\dot{F})(I, X) = (FI, (R\theta_I)_*(\eta_{FRI})_*(\dot{F}X)).$$

Recall that θ_I is the adjoint mate of $\langle F\pi_1, F\pi_2\rangle$, so $\theta_I = \epsilon_{FI}\circ L\langle F\pi_1, F\pi_2\rangle$. Now by the following calculation:

$$\begin{aligned}
R\theta_I\circ\eta_{FRI} &= R(\epsilon_{FI}\circ L\langle F\pi_1, F\pi_2\rangle)\circ\eta_{FRI} = R\epsilon_{FI}\circ RL\langle F\pi_1, F\pi_2\rangle\circ\eta_{FRI} \\
&= R\epsilon_{FI}\circ\eta_{RFI}\circ\langle F\pi_1, F\pi_2\rangle = \langle F\pi_1, F\pi_2\rangle
\end{aligned}$$

we have $W(\dot{F})(I, X) = (FI, \langle F\pi_1, F\pi_2\rangle_*(\dot{F}X))$. $\square$

5.5 Codensity Lifting of Endofunctors

We next see a lifting method using the fibration $\pi\circ- : [\mathbb{E},\mathbb{E}] \to [\mathbb{E},\mathbf{Set}]$ (Theorem 4.2). This method is a generalization of the *codensity lifting of monads* [21, Proposition 10] to endofunctors. We demonstrate that it subsumes the Kantorovich lifting in [5]. Coinductive invariants with respect to codensity liftings have a game-theoretic characterization; see [25] for details.

Let F be an endofunctor on $\mathbf{Set}$. We take the category $\mathbf{Alg}(F)$ of F-algebras and the associated forgetful functor $U : \mathbf{Alg}(F) \to \mathbf{Set}$. It comes with a natural transformation $\alpha : F\circ U \to U$, whose components are defined by the F-algebra structure: $\alpha_{(A,a)} = a : FA \to A$.

The codensity lifting of F is defined with respect to a *lifting parameter* for F, which is a pair (R,S) of functors from a discrete category $\mathbb{A}$ such that $\pi\circ S = U\circ R$:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "S"] \arrow[d, "R"'] & \mathbb{E} \arrow[d, "\pi"] \\ \mathbf{Alg}(F) \arrow[r, "U"'] & \mathbf{Set} \end{tikzcd} \tag{8}$$

Then the codensity lifting $F^{[R,S]}$ of F with respect to the above lifting parameter (R,S) is defined by the following fibered meet:

$$F^{[R,S]}X = \bigwedge_{A\in\mathbb{A},\, f\in\mathbb{E}(X,SA)} (\alpha_{RA}\circ F\pi f)^*(SA).$$

Example 5.3. *Fix a bound $b \in (0,\infty]$ for metrics. We show that the Kantorovich lifting in [5] is a codensity lifting along the $\mathbf{CLat}_\wedge$-fibration $\pi_{\mathbf{PMet}_b} : \mathbf{PMet}_b \to \mathbf{Set}$.*

Let $\alpha : F[0,b] \to [0,b]$ be an F-algebra (called an evaluation function *in [5]). We then form the following lifting parameter: $\mathbb{A} = 1$, $R = ([0,b],\alpha)$, and $S = ([0,b],d_e)$, where d_e is the standard Euclidean distance $d_e(x,y) = |x-y|$ on $[0,b]$. Then the codensity lifting with this parameter yields the following construction of pseudometric:*

$$\begin{aligned} &F^{[R,S]}(I,r) = (FI,r') \\ &\qquad r'(x,y) = \sup\left\{ |\alpha((F\pi f)(x)) - \alpha((F\pi f)(y))| \mid f \in \mathbf{PMet}_b((I,r),S) \right\}; \end{aligned}$$

note that the above sup corresponds to the meet in the fiber (Example 4.2). This is exactly the Kantorovich lifting in [5, Definition 3.1].

The codensity lifting can be characterized as a pullback when the *codensity monad* $\mathbf{Ran}_S S$ exists. Suppose that $\mathbf{Ran}_S S$ exists. Since the $\mathbf{CLat}_\wedge$-fibration $\pi : \mathbb{E} \to \mathbf{Set}$ preserves all limits, $\pi\circ\mathbf{Ran}_S S$ is a right Kan extension of $\pi\circ S$ along S. We then take the mate of the natural transformation $\alpha\circ R : F\circ\pi\circ S \to \pi\circ S$ with the right Kan extension of $\pi\circ S$ along S, and obtain $\overline{\alpha\circ R} : F\circ\pi \to \pi\circ\mathbf{Ran}_S S$.

Theorem 5.3. *Suppose that $\mathbf{Ran}_S S$ exists. Then $F^{[R,S]}$ is given by the following pullback in the* *fibration* $[\mathbb{E}, p] : [\mathbb{E}, \mathbb{E}] \to [\mathbb{E}, \mathbf{Set}]$:

$$\begin{tikzcd} F^{[R,S]} \arrow[r, dotted] & \mathbf{Ran}_S S & {[\mathbb{E},\mathbb{E}]} \arrow[d, "\pi\circ-"] \\ F\circ\pi \arrow[r, "\overline{\alpha\circ R}"'] & \pi\circ\mathbf{Ran}_S S & {[\mathbb{E},\mathbf{Set}]} \end{tikzcd}$$

From this characterization of codensity liftings, they have the following universal property. Let $\dot{F}$ be an endolifting of F along π such that $\overline{\alpha\circ R} : \dot{F} \dot{\to} \mathbf{Ran}_S S$ holds in the partial order fibration l_π given in (4). This condition is equivalent to $\alpha \circ R : \dot{F} \circ S \dot{\to} S$, which is again equivalent to: for any $A \in \mathbb{A}$, there is a (unique) $\dot{F}$-algebra $\dot{\alpha}_A : \dot{F}SA \to SA$ above the F-algebra $\alpha_{RA} : FURA \to URA$. Therefore giving such a lifting is equivalent to *lifting* the F-algebra α_{RA} on URA to the $\dot{F}$-algebra on SA. Then the codensity lifting $F^{[R,S]}$ is the *largest one* among such liftings in the fiber partial order $\mathbf{Lift}(\pi)_F$.

5.6 Lifting by Enriched Left Kan Extensions

Let Q be a commutative unital quantale, regarded as a complete and cocomplete symmetric monoidal closed category. The category Q-**Cat** of Q-enriched small categories is symmetric monoidal closed (Example 4.2). In [4], Balan et al. use Q-**Cat**-enriched left Kan extensions to derive endoliftings of endofunctors. In this section we generalize their construction to arbitrary $\mathbf{CLat}_\wedge$-fibration $\pi : \mathbb{E} \to \mathbf{Set}$ using the symmetric monoidal closed structure on $\mathbb{E}$ (Section 4.2). We omit the theory of enriched categories, but we refer Kelly's textbook [22] for the definitions and concepts in enriched category theory.

To discuss endoliftings of endofunctors, we first introduce some $\mathbb{E}$-enriched categories and $\mathbb{E}$-enriched functors, illustrated on the case where $\mathbb{E}$ is the category **Pre** of preorders and monotone functions, and π is the forgetful functor from **Pre**.

- Since $\mathbb{E}$ is symmetric monoidal closed, we view $\mathbb{E}$ as an $\mathbb{E}$-enriched category [22, Section 1.6], which is denoted by $\mathbb{E}^e$ in this paper. The hom-object of $\mathbb{E}^e$ is given by $\mathbb{E}^e(X, Y) = X \multimap Y$. When $\mathbb{E} = \mathbf{Pre}$, $\mathbf{Pre}^e$ is obtained by ordering monotone functions in a point-wise manner.

- Since the left adjoint $\Delta : \mathbf{Set} \to \mathbb{E}$ of π (see Section 4.2) is strict monoidal, it yields the *change-of-base*[5] 2-functor $\Delta_* : \mathbf{CAT} \to \mathbb{E}\text{-}\mathbf{CAT}$ [22, p.3], whose codomain is the 2-category of $\mathbb{E}$-enriched categories, $\mathbb{E}$-enriched functors and $\mathbb{E}$-enriched natural transformations [22, Section 1.2]. Δ_* maps a locally small category $\mathbb{C}$ to the $\mathbb{E}$-enriched category $\Delta_*\mathbb{C}$ defined by

$$\mathbf{Obj}(\Delta_*\mathbb{C}) = \mathbf{Obj}(\mathbb{C}), \quad (\Delta_*\mathbb{C})(I, J) = \Delta(\mathbb{C}(I, J)).$$

[5]This is a different concept from change-of-base of fibration in Section 4.1.

When $\mathbb{E} = \mathbf{Pre}$, Δ maps a set to the discrete order on this set. Δ_* then maps a category to itself, where the Hom-sets are considered as discrete orders.

- The fibration $\pi : \mathbb{E} \to \mathbf{Set}$ is also strict symmetric monoidal and naturally isomorphic to $\mathbb{E}(\Delta 1, -)$. Hence the change-of-base 2-functor $\pi_* : \mathbb{E}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ is isomorphic to the underlying category 2-functor $(-)_0 : \mathbb{E}\text{-}\mathbf{CAT} \to \mathbf{CAT}$ [22, Section 1.3]. Therefore, instead of $(-)_0$, we use π_* to extract ordinary categories (resp. functors) from $\mathbb{E}$-enriched categories (resp. functors).

- For any functor $G : \mathbb{C} \to \mathbb{E}$, we define the $\mathbb{E}$-enriched functor $\underline{G} : \Delta_*\mathbb{C} \to \mathbb{E}^e$ by

$$\underline{G}I = GI, \quad \underline{G}_{I,J} = \underline{G_{I,J}} : (\Delta_*\mathbb{C})(I,J) \to \mathbb{E}^e(GI, GJ);$$

the latter is the mate of the morphism part of functor G by the adjunction $\Delta \dashv \pi$:

$$\frac{G_{I,J} : \mathbb{C}(I,J) \to \mathbb{E}(GI, GJ) = \pi(GI \multimap GJ) = \pi(\mathbb{E}^e(GI, GJ))}{\underline{G_{I,J}} : (\Delta_*\mathbb{C})(I,J) = \Delta(\mathbb{C}(I,J)) \to \mathbb{E}^e(GI, GJ)}.$$

When $\mathbb{E} = \mathbf{Pre}$, the mate $\underline{G}_{I,J}$ is $G_{I,J}$ considered as a monotone function: indeed, since $(\Delta_*\mathbb{C})(I,J)$ is the set $\mathbb{C}(I,J)$ equipped with the discrete order, $G_{I,J}$ is automatically monotone. We also note that $\pi_*\underline{G} = G$.

The following is a generalization of [4, Theorem 3.8 and Proposition 3.43].

Theorem 5.4. *Let F be an endofunctor on* $\mathbf{Set}$ *and $C : \mathbf{Set} \to \mathbb{E}$ be a functor such that $\pi \circ C = F$ (see left triangle below). Then there is an enriched left Kan extension $\dot{F}$ of $\underline{C} : \Delta_*\mathbf{Set} \to \mathbb{E}^e$ along $\underline{\Delta} : \Delta_*\mathbf{Set} \to \mathbb{E}^e$ (see right triangle below) such that its underlying functor $\dot{F}_0 : \mathbb{E} \to \mathbb{E}$ is an endolifting of F along π.*

$$\begin{tikzcd} \mathbb{E} & & \mathbb{E} \arrow[d, "\pi"] \\ \mathbf{Set} \arrow[u, "\Delta"] \arrow[rru, "C"] \arrow[rr, "F"'] & & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{E}^e \arrow[rr, "\dot{F}"] & & \mathbb{E}^e \\ \Delta_*\mathbf{Set} \arrow[u, "\underline{\Delta}"] \arrow[rru, "\underline{C}"'] & & \end{tikzcd}$$

Proof. Since the codomain $\mathbb{E}^e$ of $\underline{C}$ has $\mathbb{E}$-enriched copowers, the enriched left Kan extension can be computed by the enriched coend:

$$\mathbf{Lan}_{\underline{\Delta}}\underline{C}X = \int^{I \in \Delta_*\mathbf{Set}} \mathbb{E}^e(\underline{\Delta}I, X) \otimes \underline{C}I;$$

see [22, (4.25)]. To simplify notation, we define an $\mathbb{E}$-enriched functor $B(I,J) : (\Delta_*\mathbf{Set})^{\mathrm{op}} \otimes \Delta_*\mathbf{Set} \to \mathbb{E}^e$ to be the body of this coend, that is, $B(I,J) \triangleq \mathbb{E}^e(\underline{\Delta}I, X) \otimes \underline{C}J$. We also define an ordinary functor $\overline{B} : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbb{E}$ by $\overline{B} \triangleq \pi_* B$. The functor $\overline{B}$ acts on objects and morphisms as follows:

$$\overline{B}(I,J) = B(I,J), \qquad\qquad \overline{B}(f,g) = (\pi B_{(I,J),(I',J')})(f,g)$$

where $B_{(I,J),(I',J')} : (\Delta_*\mathbf{Set})(I',I) \otimes (\Delta_*\mathbf{Set})(J,J') \to \mathbb{E}^e(B(I,J), B(I',J'))$ is the morphism part of B. By a simple calculation, $\overline{B}$ is equal to the functor $\lambda(I,J)\,.\,(\Delta I \multimap X) \otimes CJ$.

Because the codomain of B is $\mathbb{E}^e$, the enriched coend can be computed as an ordinary colimit of the following large diagram in $\mathbb{E}$ [22, Section 2.1]:

$$\begin{tikzcd}
\cdots & & B(I,I) & & B(J,J) & & \cdots \\
& \cdots \arrow[ul] \arrow[ur] & & \Delta_*\mathbf{Set}(J,I) \otimes B(I,J) \arrow[ul, "l_{I,J}"'] \arrow[ur, "r_{I,J}"] & & \cdots \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{9}$$

where I, J ranges over all objects in $\mathbf{Set}$, and $l_{I,J}$ and $r_{I,J}$ are respectively the uncurrying of

$$\begin{aligned}
B(I,-)_{J,I} &: \Delta_*\mathbf{Set}(J,I) \to \mathbb{E}^e(B(I,J), B(I,I)) \\
B(-,J)_{I,J} &: (\Delta_*\mathbf{Set})^{\mathrm{op}}(I,J) \to \mathbb{E}^e(B(I,J), B(J,J)).
\end{aligned}$$

in $\mathbb{E}$, respectively. Next, observe that $\Delta I \otimes X$ is an (ordinary) copower of X with $I \in \mathbf{Set}$ in $\mathbb{E}$, because

$$\mathbb{E}(\Delta I \otimes X, Y) \simeq \mathbb{E}(\Delta I, X \multimap Y) \simeq \mathbf{Set}(I, \pi(X \multimap Y)) = \mathbf{Set}(I, \mathbb{E}(X,Y)).$$

We name the passage from right to left ϕ. The bottom hom-objects of diagram (9) are thus copowers of $B(I,J)$ with $\mathbf{Set}(J,I)$ for each $I, J \in \mathbf{Set}$, and moreover, by easy calculation, we have $l_{I,J} = \phi(\overline{B}(I,-)_{J,I})$ and $r_{I,J} = \phi(\overline{B}(-,J)_{I,J})$. Therefore a colimit of the diagram (9) can be computed as an ordinary *coend* of $\overline{B} = \lambda(I,J)\,.\,(\Delta I \multimap X) \otimes CJ$.

To compute this large coend of $\overline{B}$, it suffices to show that the coend of $\pi\overline{B}$ exists in $\mathbf{Set}$, because π uniquely lifts coends. We have a natural isomorphism

$$\iota_{I,J} : \pi\overline{B}(I,J) = \pi((\Delta I \multimap X) \otimes CJ) = \mathbb{E}(\Delta I, X) \times FJ \xrightarrow{\cong} \mathbf{Set}(I, \pi X) \times FJ,$$

and the right hand side has a coend $\{i_I : \mathbf{Set}(I, \pi X) \times FI \to F\pi X\}_{I \in \mathbf{Set}}$ defined by $i_I(f,x) \triangleq Ffx$. Since π uniquely lifts colimits (Section 4.2), we obtain a coend of $\overline{B}$. To summarize, the $\mathbb{E}$-enriched left Kan extension can be computed as

$$\dot{F}X = \mathbf{Lan}_{\underline{\Delta}}\underline{C}X = \bigvee_{I \in \mathbf{Set}} (i_I \circ \iota_{I,I})_*((\Delta I \multimap X) \otimes CI).$$

When we view $\dot{F}$ as an ordinary functor $\dot{F}_0$, we have $\pi(\dot{F}_0 X) = F(\pi X)$; therefore $\dot{F}_0$ is an endolifting of F along π. $\square$

Example 5.4. *Let π be the forgetful functor from* **Pre**, *which is a* $\mathbf{CLat}_\wedge$*-fibration, and F be an endofunctor on* **Set**. *We compute the enriched left Kan extension* $\mathbf{Lan}_{\underline{\Delta}}\underline{\Delta F}$. *For* $(X, \leq) \in \mathbf{Pre}$, *the enriched left Kan extension* $\mathbf{Lan}_{\underline{\Delta}}\underline{\Delta F}(X, \leq_X)$ *is the preorder on FX generated from the following binary relation:*

$$\begin{aligned}&\{(Ffa, Fga) \mid I \in \mathbf{Set}, a \in FI, f, g \in \mathbf{Set}(I, X), \forall i \in I \,.\, fi \leq_X gi\}\\ &= \{(Fp_1 a, Fp_2 a) \mid a \in F(\leq_X)\}\end{aligned}$$

where $p_i : (\leq_X) \to X$ is the composite of the inclusion $(\leq_X) \hookrightarrow X \times X$ of the preorder relation and the projection function $\pi_i : X \times X \to X$.

When F is the powerset functor P, the enriched left Kan extension $\mathbf{Lan}_{\underline{\Delta}}\underline{\Delta P}(X, \leq_X)$ *gives the* Egli-Milner *preorder $\sqsubseteq_X$ on PX, as computed in [4, Remark 3.38]:*

$$V \sqsubseteq_X W \iff (\forall v \in V \,.\, \exists w \in W \,.\, v \leq_X w) \text{ and } (\forall w \in W \,.\, \exists v \in V \,.\, v \leq_X w).$$

6 The Category of Endoliftings

In this section, we turn to our goal of systematically *comparing* coinductive invariants. Our main result (Proposition 6.1) gives conditions under which a functor between total categories preserves coinductive invariants. We then show that the specialization preorder functor and the binary valuation truncation functor satisfy the conditions of this result, thus confirming the hypothesis advanced in Section 3.

Definition 6.1. *Let F be an endofunctor on* **Set**. *We define the category* $\mathbf{ELift}(F)$ *by the following data:*

- *An object is a pair of a* $\mathbf{CLat}_\wedge$*-fibration $\pi : \mathbb{E} \to \mathbf{Set}$ and an endolifting $\dot{F}$ of F along π.*
- *A morphism from $(\pi, \dot{F})$ to $(\rho, \ddot{F})$ is a functor $H : \mathrm{dom}(\pi) \to \mathrm{dom}(\rho)$ such that H is a lifting of* $\mathrm{Id}_{\mathbf{Set}}$ *(that is, $\rho \circ H = \pi$), and $H \circ \dot{F} = \ddot{F} \circ H$. Such H is called an* endolifting morphism.

We will sometimes suppress the fibrations and write the endolifting morphism $H : \dot{F} \to \ddot{F}$ or say "H is an endolifting morphism from $\dot{F}$ to $\ddot{F}$." Endolifting morphisms are a useful abstraction for comparing coinductive invariants of different endoliftings, thanks to the following result.

Proposition 6.1. *Let H be an endolifting morphism from $(\pi, \dot{F})$ to $(\rho, \ddot{F})$. If it preserves cartesian morphisms and fibered meets, then $H(I, \nu\dot{F}_{(I,f)}) = (I, \nu\ddot{F}_{(I,f)})$ for all F-coalgebras (I, f).*

Proof. Let $\mathbb{E} = \mathrm{dom}(\pi)$ and $\mathbb{F} = \mathrm{dom}(\rho)$. Note that H sends the final sequence in the fiber $\mathbb{E}_I$ to the final sequence in the fiber $\mathbb{F}_I$: preservation of meets ensures $H\top_{\mathbb{E}_I} = \top_{\mathbb{F}_I}$, and

$H(f^*\dot{F}(A_I)) = f^*\ddot{F}(H(A_I))$ for all $A_I \in \mathbb{E}_I$, since H is a fibration map and an endolifting morphism. Finally, H preserving meets ensures H preserves the limit of this final sequence and hence will send the $\dot{F}$-coinductive invariant for (I, f) to the $\ddot{F}$-coinductive invariant for (I, f). □

The fact that most of the conditions in this proposition do not depend on the particular source and target endoliftings makes it highly reusable. The functor H being a lifting of $\mathrm{Id}_{\mathbf{Set}}$, preserving cartesian morphisms and fibered meets are all properties independent of $\dot{F}$ and $\ddot{F}$. Therefore, after these facts are established, we can rapidly match up pairs of liftings since we only need to check the commutation condition $H \circ \dot{F} = \ddot{F} \circ H$ for each pair.

6.1 Standard Topological Endoliftings

We first prove the hypothesis advanced in Section 3 that the specialization preorder of topological invariants matches the relational invariants created by standard liftings. For this, we first need to extend the codomain of the usual specialization preorder to **ERel**.

Definition 6.2. *The* specialization preorder functor $\mathrm{Spec} : \mathbf{Top} \to \mathbf{ERel}$ *is the composition of* $S : \mathbf{Top} \to \mathbf{Pre}$ *and* $\iota : \mathbf{Pre} \to \mathbf{ERel}$*, where* S *sends a topological space to its specialization preorder and* ι *is the inclusion of preorders into endorelations.*

The "usual" specialization preorder functor S has many well-known properties, such as the fact that it has both left and right adjoints. Its left adjoint takes a preorder to its *specialization* or *Alexandroff* topology, and its right adjoint takes it to its *upper* topology.

Lemma 6.1. *The specialization preorder functor* $\mathrm{Spec} : \mathbf{Top} \to \mathbf{ERel}$ *is a fibration map from* $\pi_{\mathbf{Top}}$ *to* $\pi_{\mathbf{ERel}}$ *preserving fibered meets.*

Proof. To check it is a fibration map, use the fact that cartesian morphisms in **Top** are precisely the continuous maps of the form $f : (I, f^*\sigma) \to (J, \sigma)$ and the cartesian morphisms in **ERel** are exactly the functions preserving *and reflecting* the source relation. That Spec preserves cartesian maps then follows from the definitions.

To check it preserves fibered meets, we note Spec is a right adjoint since both S and U are right adjoints. Thus it preserves all limits, including fibered meets. □

Proposition 6.2. *Suppose* P *is a finitary polynomial in* **Set**, (I, f) *is a* $\mathbf{Set}[\![P]\!]$*-coalgebra, and* $\dot{P}$ *is a finitary polynomial in* **Top** *satisfying* $\mathbf{P}(\pi_{\mathbf{Top}})(\dot{P}) = P$. *Let* $\dot{F}$ *be the interpretation of* $\dot{P}$ *in* **Top**, *and let* $\ddot{F}$ *be the interpretation of* $\mathbf{P}(\mathrm{Spec})(\dot{P})$ *in* **ERel**. *Then* $\mathrm{Spec}(\nu\dot{F}_{(I,f)}) = \nu\ddot{F}_{(I,f)}$.

Proof. First, we show Spec is an endolifting morphism from $\dot{F}$ to $\ddot{F}$. Clearly, Spec is a lifting of $\mathrm{Id}_{\mathbf{Set}}$. Since Spec preserves products and coproducts, Proposition 5.1 shows the commutation condition $\mathrm{Spec} \circ \dot{F} = \ddot{F} \circ \mathrm{Spec}$.

Lemma 6.1 supplies the remaining conditions of Proposition 6.1 in this context. □

This proposition is the key result that allows us to compare topological coinductive invariants with relational coinductive invariants via the specialization order, as we illustrated in Section 3. In those examples, we were considering the polynomial $(2, s) \times \mathrm{Id} \times \mathrm{Id}$ in **Top**, whose image under $\mathbf{P}(\mathrm{Spec})$ is $(2, \leq_2) \times \mathrm{Id} \times \mathrm{Id}$ as a polynomial in **ERel**. Hence, the coinductive invariants created by the interpretations of these polynomials as endofunctors are matched up by Spec.

In particular, when the constants in the polynomial are restricted so that their specialization order is the diagonal relation, we obtain topological coinductive invariants whose specialization order is bisimilarity.

Corollary 6.1. *Suppose P is a finitary polynomial in* **Set** *and $\dot{P}$ is a finitary polynomial in* **Top** *with the property that* $\mathbf{P}(\mathrm{Spec})(\dot{P}) = \mathbf{Rel}(P)$. *Then define three endofunctors on* **Set**, **Top**, *and* **ERel***:*

$$F = \mathbf{Set}[\![P]\!], \quad \dot{F} = \mathbf{Top}[\![\dot{P}]\!], \quad \ddot{F} = \mathbf{ERel}[\![\mathbf{Rel}(P)]\!].$$

Note that $\dot{F}, \ddot{F}$ are respectively **Top**- *and* **ERel**-*liftings of F by Corollary 5.1. Suppose that (I, f) is an F-coalgebra.*

1. $\mathrm{Spec}(\nu\dot{F}_{(I,f)}) = \nu\ddot{F}_{(I,f)}$. *(Recall the $\ddot{F}$-coinductive invariant is the bisimilarity relation.)*

2. *Two points are topologically indistinguishable in $\nu\dot{F}_{(I,f)}$ if and only if they are bisimilar.*

3. *$\nu\dot{F}_{(I,f)}$ is an R_0 topology (topologically indistinguishable points each have a neighborhood not containing the other).*

4. *(I, f) is simple (has no proper quotients) if and only if $\nu\dot{F}_{(I,f)}$ is a T_1 topology (distinct points each have a neighborhood not containing the other).*

Proof.

1. This is a special case of Proposition 6.2.

2. Two points are topologically indistinguishable if and only if they are equivalent in the specialization preorder, which we have shown is bisimilarity.

3. For finitary polynomial endofunctors on **Set**, bisimilarity is an equivalence relation. A topology's specialization preorder is an equivalence relation if and only if the topology is R_0.

4. For finitary polynomial endofunctors on **Set**, a coalgebra being simple is equivalent to bisimilarity being the diagonal relation. A topology's specialization preorder is diagonal if and only if the topology is T_1.

□

Note that though this proposition requires the specialization order of the constants in a lifting to be the diagonal, the specialization order of the induced topology need not be the diagonal.

6.2 BVal Endoliftings

Next, we consider coinductive invariants in the category **BVal**, which are also called behavioural (pseudo)metrics when they satisfy the necessary conditions. A common desiderata is that the kernel of the behavioural metric (i.e., the relation consisting of points at distance 0) is the bisimilarity relation. We can prove this statement with endolifting morphisms.

Definition 6.3. *The* truncation functor (at $\epsilon \in \mathbb{R}^+$)*, denoted* $T_\epsilon : \mathbf{BVal} \to \mathbf{ERel}$ *acts on objects by* $(I, r) \mapsto (I, \{(i, i') \in I \times I \mid r(i, i') \leq \epsilon\})$ *and sends the non-expansive map* f *to the relation-preserving map* f*.*

The functor $L_\epsilon : \mathbf{ERel} \to \mathbf{BVal}$ *acts on objects by* $(I, R) \mapsto (I, L_\epsilon(R))$*, where* $L_\epsilon(R)(i, i') = 0$ *if* $(i, i') \in R$ *and is* ϵ *otherwise. It also has the trivial action on functions.*

This action on morphisms obviously preserves identities and composition, but it takes a quick check to verify that nonexpansive maps are indeed sent to relation-preserving maps by T_ϵ and vice-versa for L_ϵ.

Lemma 6.2. *For each* $\epsilon \in \mathbb{R}^+$*, the functor* $T_\epsilon : \mathbf{BVal} \to \mathbf{ERel}$ *is a fibration map from* $\pi_{\mathbf{BVal}}$ *to* $\pi_{\mathbf{ERel}}$ *preserving fibered meets.*

Proof. To check it is a fibration map, use the facts that cartesian morphisms in **BVal** are isometries and cartesian morphisms in **ERel** preserve and reflect their source relations. Then checking that T_ϵ preserves cartesian morphisms is straightforward.

The left adjoint of T_ϵ is L_ϵ. □

Proposition 6.3. *Suppose* $\dot{P}$ *is a finitary polynomial in* **BVal** *and define* $\dot{F} = \mathbf{BVal}[\![\dot{P}]\!]$ *and* $\ddot{F} = \mathbf{ERel}[\![\mathbf{P}(T_\epsilon)(\dot{P})]\!]$*. Then* T_ϵ *is an endolifting morphism from* $\dot{F}$ *to* $\ddot{F}$*.*

Proof. That T_ϵ is a lifting of $\mathrm{Id}_{\mathbf{Set}}$ is clear. We verify the commutation condition by again invoking Proposition 5.1. Since T_ϵ is a right adjoint, it preserves products. The only remaining obligation is to check preservation of coproducts, which is a routine exercise. □

Corollary 6.2. *Suppose* $\dot{P}$ *is a finitary polynomial in* **BVal***. Let* $F = \mathbf{Set}[\![\mathbf{P}(\pi_{\mathbf{BVal}})(\dot{P})]\!]$*,* $\dot{F} = \mathbf{BVal}[\![\dot{P}]\!]$*, and* $\ddot{F} = \mathbf{ERel}[\![\mathbf{P}(T_\epsilon)(\dot{P})]\!]$*. If* (I, f) *is an* F*-coalgebra, then* $T_\epsilon(\nu \dot{F}_{(I,f)}) = \nu \ddot{F}_{(I,f)}$*.*

If additionally $\epsilon = 0$ *and* $\mathbf{P}(T_0)(\dot{P}) = \mathbf{Rel}(\mathbf{P}(\pi_{\mathbf{BVal}})(\dot{P}))$*, then the image of a behavioural pseudometric under* T_0 *(the kernel of that pseudometric) is bisimilarity.*

Proof. Proposition 6.3 and Lemma 6.2 establish the conditions of Proposition 6.1 for T_ϵ.

In the special case, $\ddot{F}$ is a canonical relation lifting and hence $\nu \ddot{F}_{(I,f)}$ is bisimilarity. □

This establishes T_0 sends behavioural metrics on finitary polynomial endofunctors to bisimilarity. We also want to show that the Hausdorff lifting of P_{fin} has bisimilarity as its kernel. Therefore, we extend our induction on the structure of the finitary polynomial to include pushforwards along natural transformations. In general, T_ϵ being an endolifting morphism between two functors does not imply that it is an endolifting morphism between their pushforwards. However, we can add a condition under which this does hold.

Proposition 6.4. *Let F, G be endofunctors on* **Set**, *$T_\epsilon : (\pi_{\mathbf{BVal}}, \dot{F}) \to (\pi_{\mathbf{ERel}}, \ddot{F})$ be a morphism in $\mathbf{ELift}(F)$, and $\tau : F \to G$ be a natural transformation, and $\dot{G}$ and $\ddot{G}$ be pushforwards of $\dot{F}$ and $\ddot{F}$ along τ. Further suppose for every set I, every $f, f' \in GI$ and $r : I \times I \to \mathbb{R}^+$, the greatest lower bound for $\{\dot{F}r(p, p') \mid \tau p = f \text{ and } \tau p' = f'\}$ is achieved in this set. Then T_ϵ is an endolifting morphism from $(\pi, \dot{G})$ to $(\rho, \ddot{G})$.*

Proof. We prove that $(T_\epsilon \circ \dot{G})(I, r) = (\ddot{G} \circ T_\epsilon)(I, r)$ for all I and r.

We know the pushforward operation in **BVal** and **ERel** explicitly, so:

$$\dot{G}(I, r) = (GI, \lambda f f'. \inf_{\substack{\tau p = f \\ \tau p' = f'}} \dot{F}r(p, p'))$$

$$\ddot{G}(I, R) = (GI, \{(f, f') \mid \exists p, p' \in FI. \tau p = f, \tau p' = f' \text{ and } (p, p') \in \ddot{F}R\})$$

Given particular f and f', our assumption about the lower bound ensures there are $p_1, p_2 \in FI$ with $\tau p_1 = f$ and $\tau p_2 = f'$ such that $\inf_{\substack{\tau p = f \\ \tau p' = f'}} \dot{F}r(p, p') = \dot{F}r(p_1, p_2)$. Therefore,

$$\begin{aligned}
(T_\epsilon \circ \dot{G})r &= \{(f, f') \mid \inf_{\substack{\tau p = f \\ \tau p' = f'}} \dot{F}r(p, p') \le \epsilon\} \\
&= \{(f, f') \mid \exists p_1, p_2. \tau p_1 = f, \tau p_2 = f' \text{ and } \dot{F}r(p_1, p_2) \le \epsilon\} \\
&= \{(f, f') \mid \exists p_1, p_2. \tau p_1 = f, \tau p_2 = f' \text{ and } (p_1, p_2) \in (T_\epsilon \circ \dot{F})r\} \\
&= \{(f, f') \mid \exists p_1, p_2. \tau p_1 = f, \tau p_2 = f' \text{ and } (p_1, p_2) \in (\ddot{F} \circ T_\epsilon)r\} \\
&= (\ddot{G} \circ T_\epsilon)(r)
\end{aligned}$$

as desired. □

We can now apply this proposition to obtain the following corollary.

Corollary 6.3. *T_0 is an endolifting morphism from $(\pi_{\mathbf{BVal}}, P_{\text{fin}}^{\mathbf{BVal}})$ to $(\pi_{\mathbf{ERel}}, P_{\text{fin}}^{\mathbf{ERel}})$. Therefore, the Hausdorff behavioural metric on P_{fin}-coalgebras has $P_{\text{fin}}^{\mathbf{ERel}}$-coinductive invariants at its kernel.*

Proof. Proposition 6.3 implies T_0 is an endolifting morphism from the standard **BVal** lifting for the list functor to the standard **ERel** lifting for the list functor. We know $P_{\text{fin}}^{\mathbf{BVal}}$ and $P_{\text{fin}}^{\mathbf{ERel}}$ are the pushforwards of these list functors along set_X in their respective total categories. Hence to apply Proposition 6.4 we only need to show the infimum from $\mathcal{H}'d(K, L)$ is always achieved, which we exhibited is true with $s(K, L)$ and $t(K, L)$ from section 5.3.

Since T_0 is an endolifting morphism, we can use Lemma 6.2 to conclude the Hausdorff behavioural metric has bisimilarity at its kernel. □

6.3 Approximate Bisimulations: An Example From Control Theory

Here we present an example from a rather different context: *approximate bisimulation* by Girard and Pappas [12]. Defined as a binary relation on a metric space that is subject to the "mimicking" condition, the notion is widely used in control theory as a quantitative relaxation of usual (Milner-Park) bisimulation that allows bounded errors. Its principal use is in bounding errors caused by some abstraction of dynamical systems: given the original dynamics $\mathcal{S}$, one derives its abstraction $\mathcal{A}$; by exhibiting an *ϵ-approximate bisimulation* between $\mathcal{S}$ and $\mathcal{A}$, one then shows that the difference between the trajectory of $\mathcal{A}$ and that of $\mathcal{S}$ is bounded by ϵ. Such abstraction methods include: state space discretization [14] and ignoring switching delays [24]. See [13] for an overview.

In the above scenario, an ϵ-approximate bisimulation between $\mathcal{S}$ and $\mathcal{A}$ is synthesized through analysis of the continuous dynamics of $\mathcal{S}$: for example the *incremental stability* of $\mathcal{S}$ yields an approximate bisimulation via its Lyapunov-type witness. Another common strategy for finding an approximate bisimulation is via a *bisimulation function*. Our goal here is to describe the latter strategy in the current coalgebraic and fibrational framework.

We fix the set O of output values together with a distance function $d : O \times O \to \mathbb{R}^+$, and a U-labeled finitely branching transition system $(Q, \delta : Q \to \prod_{u \in U} P_{\mathrm{fin}} Q)$ with an output function $o : Q \to O$. An *ϵ-approximate bisimulation relation* is a binary relation $R \subseteq Q \times Q$ such that

$$\begin{aligned}
\forall (q, q') \in R \,.\, d(o(q), o(q')) \le \epsilon \text{ and } \forall l \in U \,. \\
&(\forall r \in Q \,.\, r \in \delta(l, q) \implies \exists r' \in Q \,.\, r' \in \delta(l, q') \text{ and } (r, r') \in R) \text{and} \\
&(\forall r' \in Q \,.\, r' \in \delta(l, q') \implies \exists r \in Q \,.\, r \in \delta(l, q) \text{ and } (r, r') \in R). \qquad (10)
\end{aligned}$$

The difference from the usual Milner-Park bisimulation is that R is additionally required to witness the ϵ-proximity of outputs of related states q and q'.

A bisimulation function is a quantitative (real-valued) witness for an approximate bisimulation. In many settings in control theory where dynamics are smooth and described by ordinary differential equations, such real-valued functions are easier to come up with than an approximate bisimulation itself. For the above LTS, a function $v : Q \times Q \to \mathbb{R}^+$ is a *bisimulation function* if it satisfies, for each $q, q' \in Q$,

$$\max\Big(d\big(o(q), o(q')\big), \sup_{l \in U} \mathcal{H} v\big(\delta(l, q), \delta(l, q')\big) \Big) \le v(q, q') \qquad (11)$$

A crucial fact is that a bisimulation function v gives rise to an ϵ-approximate bisimulation $\{(q, q') \mid v(q, q') \le \epsilon\}$. See generally [13]. Below we give a coalgebraic account of this argument. We introduce three endofunctors on **BVal**, **ERel** and **Set** ($\epsilon \in \mathbb{R}^+$):

$$\dot{F}_{\mathrm{lts}} \triangleq (O, d) \times \prod_{u \in U} P^{\mathbf{BVal}}_{\mathrm{fin}} \qquad \ddot{F}_{\epsilon\text{-lts}} \triangleq (O, T_\epsilon d) \times \prod_{u \in U} P^{\mathbf{ERel}}_{\mathrm{fin}} \qquad F_{\mathrm{lts}} \triangleq O \times \prod_{u \in U} P_{\mathrm{fin}}$$

We then package the U-labeled finitely branching transition system (Q, δ) and the output function o into a single F_{lts}-coalgebra $\mathcal{Q} = (Q, \langle o, \delta \rangle : Q \to F_{\mathrm{lts}} Q)$.

Proposition 6.5. *The following holds.*

1. *$\dot{F}_{\mathrm{lts}}$ and $\ddot{F}_{\epsilon\text{-lts}}$ are* **BVal**- *and* **ERel**-*liftings of F_{lts}, respectively.*
2. *$(Q, v) \in$ **BVal** is a $\dot{F}_{\mathrm{lts}}$-invariant on $\mathcal{Q}$ if and only if $v : Q \times Q \to \mathbb{R}^+$ is a bisimulation function.*
3. *$(Q, R) \in$ **ERel** is a $\ddot{F}_{\epsilon\text{-lts}}$-invariant on $\mathcal{Q}$ if and only if $R \subseteq Q \times Q$ is an ϵ-approximate bisimulation relation.*
4. *T_ϵ is an endolifting morphism from $(\pi_{\mathbf{BVal}}, \dot{F}_{\mathrm{lts}})$ to $(\pi_{\mathbf{ERel}}, \ddot{F}_{\epsilon\text{-lts}})$.*
5. *if $v : Q \times Q \to \mathbb{R}^+$ is a bisimulation function, then $T_\epsilon(Q, v)$ is an ϵ-approximate bisimulation.*

Proof. 1) Easy. 2,3) By unfolding the definitions the following can be observed: $\ddot{F}_{\epsilon\text{-lts}}$-invariants on Q are nothing but ϵ-approximate bisimulations; and $\dot{F}_{\mathrm{lts}}$-invariants on Q are bisimulation functions. 4) Thanks to Proposition 6.3 and Corollary 6.3, the functor T_ϵ—which sends the function $v : Q \times Q \to \mathbb{R}^+$ to the relation $\{(q, q') \mid v(q, q') \le \epsilon\}$—is an endolifting morphism from $\dot{F}_{\mathrm{lts}}$ to $\ddot{F}_{\epsilon\text{-lts}}$. 5) T_ϵ transfers a $\dot{F}_{\mathrm{lts}}$-invariant (Q, v) to a $\ddot{F}_{\epsilon\text{-lts}}$-invariant $(Q, T_\epsilon v)$ on Q, that is, a bisimulation function to an ϵ-approximate bisimulation. $\square$

7 Conclusions and Future Work

We presented a fibrational framework for various extensions of (bi)simulation notions. On the categorical side our focus has been on structural aspects of fibrations such as fibration maps and lifting by Kan extensions; on the application side we took examples from quantitative reasoning about systems. This has allowed us to capture known constructions in more abstract and general terms, such as the Hausdorff pseudometric and approximate bisimulation in control theory.

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